

An invariance principle for stochastic series I. Gaussian limits

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Abstract

We study invariance principles and convergence to a Gaussian limit for stochastic series of the form $S(c, Z) = \sum_{m=1}^{\infty} \sum_{\alpha_1 < \dots < \alpha_m} c(\alpha_1, \dots, \alpha_m) \prod_{i=1}^m Z_{\alpha_i}$ where Z_k , $k \in \mathbb{N}$ is a sequence of centred independent random variables of unit variance. In the case when the Z_k 's are Gaussian, $S(c, Z)$ is an element of the Wiener chaos and convergence to a Gaussian limit (so the corresponding nonlinear CLT) has been intensively studied by Nualart, Peccati, Nourdin and several other authors. The invariance principle consists in taking Z_k with a general law. It has also been considered in the literature, starting from the seminal papers of Jong, and a variety of applications including U -statistics are of interest. Our main contribution is to study the convergence in total variation distance and to give estimates of the error.

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1 Introduction

The aim of this paper is to provide invariance principles for stochastic series of the form

$$S_N(c, Z) = \sum_{m=1}^N \Phi_m(c, Z), \quad \text{with} \quad \Phi_m(c, Z) = \sum_{|\alpha|=m} c(\alpha) Z^\alpha. \quad (1.1)$$

Let us explain the notation: $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ is a multi-index with length $|\alpha| = m$; $Z_k, k \in \mathbb{N}$, is a sequence of independent random variables with $\mathbb{E}(Z_k) = 0$ and $\mathbb{E}(Z_k^2) = 1$ and $Z^\alpha = \prod_{i=1}^m Z_{\alpha_i}$. Moreover $c(\alpha) \in \mathbb{R}$ are coefficients which are symmetric, null on the diagonals and verify the normalization condition

$$\mathbb{E}(S_N(c, Z)^2) = \sum_{m=1}^N m! \sum_{|\alpha|=m} c^2(\alpha) = 1. \quad (1.2)$$

There are two types of results: first we consider infinite series, so $N = \infty$, and sequences of coefficients $c^{(n)} = (c^{(n)}(\alpha))_\alpha$ which verify the above normalization condition and we give sufficient conditions for the convergence of $S_\infty(c^{(n)}, Z)$ to a standard Gaussian law. This will be convergence in law on one hand and convergence in total variation distance on the other hand. These are asymptotic results. In a second stage we restrict ourself to finite series, so $N < \infty$ is fixed, and we obtain non asymptotic estimates of the error. Here our first aim is to estimate

$$\Delta_{Z, \overline{Z}}(c, f) = |\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(S_N(c, \overline{Z})))|$$

where $\overline{Z}_k, k \in \mathbb{N}$, is a sequence of independent standard normal random variables. The second aim is to estimate the distance between the law of $S_N(c, Z)$ and the standard Gaussian law. In the case $N = 1$ this is just the CLT. Notice also that since $\overline{Z}_k, k \in \mathbb{N}$, are standard normal random variables then $\Phi_m(c, \overline{Z})$ is (in law) a multiple stochastic integral of order m and, starting with the seminal paper of Nualart and Peccati [23], a lot of work has been done in order to obtain the CLT for such multiple integrals. So, once we are able to estimate $\Delta_{Z, \overline{Z}}(c, f)$ (this is the invariance principle), we may use the above mentioned results concerning the Wiener chaos, in order to obtain the distance to the Gaussian distribution. However, the two problems have to be discussed separately because the Gaussian law is not the single possible limit for such series: for example, Nourdin and Peccati in [17] give sufficient conditions in order that such series converge to a chi-squared distribution. We address the problem of non Gaussian limits in the working paper [3].

This type of nonlinear invariance principle turns out to be of interest in several very different fields of applications: Mossel, O’Donnell and Oleszkiewicz in [14] provided interesting applications in theoretical computer science and in social choice theory. And similar objects appear in the U -statistics theory see e.g. Koroljuk and Borovskich [10].

The first results concerning the convergence in law of $S_N(c, Z)$ to the Gaussian distribution has been obtained by Jong [7] and [8]. Afterwards, Mossel, O’Donnell and Oleszkiewicz in [14] obtained an invariance principle in Kolmogorov distance. Finally, under a supplementary regularity condition on the

laws of Z_k (that we discuss below) Nourdin and Poly [20] gave a convergence result in total variation distance. Let us shortly present these results. The central quantity which controls the convergence of the series $S_N(c, Z)$ is the so called “low influence factor” defined by

$$\bar{\delta}_N(c) = \sum_{m=1}^N \delta_m(c) \quad \text{with} \quad \delta_m(c) = \max_k \sum_{|\alpha|=m-1} c^2(k, \alpha). \quad (1.3)$$

Roughly speaking $\sum_{|\alpha|=m-1} c^2(k, \alpha)$ may be considered as the influence on the particle k of all the other particles. And if $\bar{\delta}_N(c)$ is small we say that we have low influence. Consider now a sequence of coefficients $c^{(n)}, n \in \mathbb{N}$ and the corresponding series $S_N(c^{(n)}, Z)$. In [14] one proves that, if $\lim_{n \rightarrow \infty} \bar{\delta}_N(c^{(n)}) = 0$ then

$$\lim_{n \rightarrow \infty} \sup_{a \in \mathbb{R}} \Delta_{Z, \bar{Z}}(c^{(n)}, 1_{(a, \infty)}) = 0$$

which means that the Kolmogorov distance between $S_N(c^{(n)}, Z)$ and $S_N(c^{(n)}, \bar{Z})$ converges to zero as $n \rightarrow \infty$. Actually the authors of that paper look to a more particular problem, namely to a single level $\Phi_m(c^{(n)}, Z)$ and $\Phi_m(c^{(n)}, \bar{Z})$, so in this sense our problem is more general because it concerns series. Moreover, in [20] for $\Phi_m(c^{(n)}, Z)$ and $\Phi_m(c^{(n)}, \bar{Z})$ as well, under the hypothesis $\lim_{n \rightarrow \infty} \bar{\delta}_N(c^{(n)}) = 0$, one proves convergence in total variation distance that is

$$\lim_{n \rightarrow \infty} \sup_{\|f\|_\infty \leq 1} \Delta_{Z, \bar{Z}}(c^{(n)}, f) = 0.$$

But the authors are obliged to assume more regularity, namely that the law of Z_k is locally lower bounded by the Lebesgue measure: there exist $r, \varepsilon > 0$ and $z_k \in \mathbb{R}$ such that for every measurable set $A \subset B_r(z_k)$ one has

$$\mathbb{P}(Z_k \in A) \geq \varepsilon \lambda(A) \quad (1.4)$$

where λ is the Lebesgue measure. (1.4) is analogous to what is known in the literature as the Doeblin condition. Then they use a splitting method and the Γ -calculus settled in [5] to obtain the regularity which is needed in order to handle test functions f which are just measurable. This strategy is close to the method that we use ourselves in this paper. Notice that the hypothesis (1.4) is in fact very mild (almost necessary): indeed, in the case of the classical CLT (which corresponds to $N = 1$), Prohorov proved in [25] that in order to obtain convergence in total variation distance one needs that the law of the random variables has at least a piece of absolutely continuous component (and it turns out that this is very close to (1.4), see the discussion in Section 2 of [1]).

Let us now present the contributions of our paper. We first prove that if $f \in C_b^3$ then

$$\Delta_{Z, \bar{Z}}(c, f) \leq C \|f'''\|_\infty \bar{\delta}_N(c) \quad (1.5)$$

where C is a constant which depends on $M_p = M_p(Z) = \max_k \mathbb{E}(|Z_k|^p)$ with $p = 3$, see Theorem 3.1 for a precise statement. In this case the regularity condition (1.4) is not required. The proof is a rather standard application of the Lindeberg method.

We discuss now the convergence to a Gaussian law. First we have to introduce the “fourth cumulant” defined for a random variable X by $\kappa_4(X) = \mathbb{E}(X^4) - 3\mathbb{E}(X^2)^2$. This quantity is known to be a measure of the distance between the law of X and the standard Gaussian law in the sense that, if $\lim_{n \rightarrow \infty} \kappa_4(X_n) = 0$ then $X_n \rightarrow G$ in law, where G is a standard normal random variable (for which $\kappa_4(G) = 0$). The celebrated “Fourth Moment Theorem” proved in [23] (and then refined in several other papers of Nualart, Nourdin, Peccati and co-authors, see <https://sites.google.com/site/malliavinstein/home> for updated references on this subject) asserts that the convergence of the multiple stochastic integrals $\Phi_m(c^{(n)}, \bar{Z})$ to the normal law is equivalent to the convergence of the fourth cumulant. So we define

$$\bar{\kappa}_N(c) = \sum_{m=1}^N \kappa_4^{1/4}(\Phi_m(c, \bar{Z})), \quad \text{with } \bar{Z}_k, k \in \mathbb{N}, \text{ i.i.d. standard normal.} \quad (1.6)$$

Notice that, having used independent standard normal random variables, $\bar{\kappa}_N(c)$ is in some sense an intrinsic quantity related to the coefficient c .

We present now our convergence results. Let $c^{(n)} = (c^{(n)}(\alpha))_\alpha$ be a sequence of coefficients which verify the normalization condition (1.2) and such that for every $N \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \bar{\kappa}_N(c^{(n)}) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \bar{\delta}_N(c^{(n)}) = 0, \quad (1.7)$$

$\bar{\delta}_N(c^{(n)})$ being given in (1.3). We will consider also the following “uniformity” assumption:

$$\limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k \geq N} k^q \times k! \sum_{|\alpha|=k} |c^{(n)}(\alpha)|^2 = 0, \quad q \in \mathbb{N}. \quad (1.8)$$

We prove that if (1.7) and (1.8) hold, with $q = 0$, then $S_\infty(c^{(n)}, Z) \rightarrow G$ in law as $n \rightarrow \infty$ (so, a result for infinite series). In the case of multiple stochastic integrals (that is when Z_k are standard normal) this result has already been proved in [9], so, what is new here, is the invariance principle we are going to introduce in (1.9) (see Theorem 3.2 for the precise statement, which needs some more hypotheses on the moments). Moreover, if (1.8) holds with $q = 1$, then we prove that $S_\infty(c^{(n)}, Z) \rightarrow G$ in total variation distance (see Theorem 6.4 for the precise statement). Notice that

$$\|S_\infty(c^{(n)}, Z)\|_2^2 = \sum_{k \geq 1} k! \sum_{|\alpha|=k} |c^{(n)}(\alpha)|^2$$

so, in some sense, (1.8) with $q = 0$ says that $S_\infty(c^{(n)}, Z)$, $n \in \mathbb{N}$, belongs to a “uniform class” in L^2 . And if (1.8) holds with $q = 1$, one gets a stronger uniformity condition concerning the Malliavin derivatives - which is morally coherent.

We come back now to our non asymptotic results. The challenging problem now is to replace $\|f'''\|_\infty$ with $\|f\|_\infty$ in (1.5), so to obtain the distance in total variation between $S_N(c, Z)$ and $S_N(c, \bar{Z})$. In Theorem 6.1 we prove that for each $p_* \geq 1$ one has

$$\Delta_{Z, \bar{Z}}(c, f) \leq C_{*N} \|f\|_\infty (\bar{\delta}_N(c) + \bar{\kappa}_N^{p_*}(c)) \quad (1.9)$$

where C_{*N} is a constant which depends on p_* , on N and on $M_p(Z)$ for some p . Here we are obliged to take a finite N .

In the papers presented above, that is [14] and [20], the only quantity which was supposed to be small is the low influence factor term $\bar{\delta}_N(c)$. So, the fact that the 4th cumulant term $\bar{\kappa}_N(c)$ appears in (1.9) may be seen as a weak point. However, as long as we deal with convergence to a Gaussian law, we know that, by the Fourth Moment Theorem, we need to ask $\bar{\kappa}_N(c^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$. For a general limit, in [3] we will prove that

$$\Delta_{Z, \bar{Z}}(c, f) \leq C_N \bar{\delta}_N^{1/N}(c). \quad (1.10)$$

The advantage of (1.10) is that $\bar{\kappa}_N(c)$ does not appear in the right hand side, so (1.10) works for general limits. But the interest of (1.9) is that we get a more accurate estimate (because one has $\bar{\delta}_N(c)$ instead of $\bar{\delta}_N^{1/N}(c)$).

Let us now present our nonlinear CLT. We set $|c|_m^2 = \sum_{|\alpha|=m} c(\alpha)^2$ and $\alpha_N(c) = \min_{m \leq N} |c|_m \mathbf{1}_{|c|_m > 0}$. One may prove (see next (2.13)) that $\bar{\delta}_N(c) \leq \alpha_N^{-1}(c) \bar{\kappa}_N(c)$, so (1.9) with $p_* = 1$ reads

$$\Delta_{Z, \bar{Z}}(c, f) \leq C_{*N} \|f\|_\infty (1 + \alpha_N^{-1}(c)) \bar{\kappa}_N(c). \quad (1.11)$$

Moreover the Fourth Moment Theorem by Nourdin and Peccati in [17] says that if \bar{Z}_k , $k \in \mathbb{N}$, are standard normal then

$$d_{TV}(S_N(c, \bar{Z}), G) \leq C_N \bar{\kappa}_N(c).$$

Therefore, putting things together, in Theorem 6.2 we prove that

$$|\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(G))| \leq C \|f\|_\infty (1 + \alpha_N^{-1}(c)) \bar{\kappa}_N(c). \quad (1.12)$$

The proof of (1.9) is based on integration by parts methodology, inspired from Malliavin calculus and which has been settled in [4], [2] and has already been used in [1]. As usual the difficult point which has to be handled is the non degeneracy condition. In [20] Nourdin and Poly use the Carbery-Wright inequality for small balls probabilities in order to solve a similar problem - and we are doing the same in [3]. This approach avoids to use the cumulants $\bar{\kappa}_N(c)$ but makes appear the power $1/N$ in $\bar{\delta}_N^{1/N}(c)$. So we give out this approach here and we use an ad-hoc method based on martingale arguments and Burkholder inequality. A serious technical difficulty comes from the fact that for general stochastic series we do not have the product formula which is available for multiple stochastic integrals (see Lemma B.4 and Remark B.5 in Appendix B).

The paper is organized as follows. In section 2 we introduce the rather heavy notation and in Section 3 we prove the convergence result for smooth test functions, that is (1.5). In Section 4 we recall the variational calculus that use here and in Section 5 we estimate the Sobolev norms of $S_N(c, Z)$ and give the non degeneracy estimate. In order to obtain this last estimate a heavy calculus based on Burkholder inequalities and a martingale method is needed - we postpone these calculations in Appendix A. Finally in Section 6 we prove the main results, namely (1.9) and (1.12).

2 Notations

The basic objects which appear in this paper are the following.

- We denote $\Gamma_m = \mathbb{N}^m$, that is Γ_m is the set of the multi-indexes $\beta = (\beta_1, \dots, \beta_m)$. When $m = 0$, we define Γ_0 as the set containing only the null multi-index $\beta = \emptyset$. For $\beta \in \Gamma_m$, we say that β has length m , and we define the length as $|\beta| = m$. We set $\Gamma = \cup_{m \geq 0} \Gamma_m$ the set of all multi-indexes. For a fixed $J \in \mathbb{N}$, we set $\Gamma_m(J)$ as the multi-indexes whose components do not exceed J : $\Gamma_m(J) = \{\beta \in \Gamma_m : \beta_i \leq J \text{ for every } i\}$. Finally we consider the set of “ordered” multi-indexes Γ_m^o and $\Gamma_m^o(J)$: when considering the superscript o we mean that the multi-index β has ordered components, that is $\beta_i < \beta_{i+1}$ for all i . For $z \in \mathbb{R}^{\mathbb{N}}$, $z = (z_k)_{k \in \mathbb{N}}$ and for $\beta \in \Gamma_m$ we denote

$$z^\beta = \prod_{k=1}^m z_{\beta_k}. \quad (2.1)$$

- We consider a sequence of independent (non necessarily identically distributed) random variables $Z_k, k \in \mathbb{N}$ which, for some $p \geq 1$, verify

$$\mathbb{E}(Z_k) = 0 \text{ and } \mathbb{E}(Z_k^2) = 1 \forall k, M_p(Z) := \max_k \mathbb{E}(|Z_k|^p) < \infty \text{ and we set } \bar{M}_p(Z) = b_p M_p(Z) \quad (2.2)$$

where b_p is the constant in the Burkholder inequality of order p (see Appendix A).

- We consider a family of coefficients $c(\alpha)$, $\alpha \in \Gamma$, which are symmetric and null on all the diagonals: if $\alpha = (\alpha_1, \dots, \alpha_m) \in \Gamma_m$, then for every permutation π of $\{1, \dots, m\}$ one has $c(\alpha) = c(\alpha_\pi)$ with $\alpha_\pi = (\alpha_{\pi_1}, \dots, \alpha_{\pi_m})$; if $\alpha_i = \alpha_j$ for some $i \neq j$ then $c(\alpha) = 0$.

We set

$$|c|_m = \left(\sum_{\alpha \in \Gamma_m} c^2(\alpha) \right)^{1/2}, \quad \|c\|_m = \left(\sum_{i=1}^m |c|_i^2 \right)^{1/2}, \quad (2.3)$$

and, for $q \in \mathbb{N}$ and $M > 0$,

$$N_q(c, M) = \left(\sum_{m=q}^{\infty} M^{m-q} \times \frac{m!}{(m-q)!} \times m! |c|_m^2 \right)^{1/2}. \quad (2.4)$$

Moreover we denote

$$\delta_1(c) = \sup_k |c(k)| \quad \text{and} \quad \delta_m(c) = \sup_k \left(\sum_{\alpha \in \Gamma_{m-1}} c^2(k, \alpha) \right)^{1/2}, \quad m \geq 2. \quad (2.5)$$

We use the notation $(k, \alpha) = (k, \alpha_1, \dots, \alpha_{m-1})$ for $\alpha = (\alpha_1, \dots, \alpha_{m-1}) \in \Gamma_{m-1}$ (note that if $m = 1$ then Γ_{m-1} contains only the void multi-index and $c^2(k, \emptyset) = c^2(k)$). Roughly speaking $\delta_m(c)$ quantifies the maximum action of a single particle on the other ones, and, if $\delta_m(c)$ is required to be small, we say that we have a “low influence” condition. Moreover we denote

$$\varepsilon_0(c, M) = \sup_k \left(\sum_{m=1}^{\infty} M^{2m} \times m! \left(\sum_{\alpha \in \Gamma_m} c^2(k, \alpha) \right) \right)^{1/2} \leq \sum_{m=0}^{\infty} M^{2m} \times m! \times \delta_{m+1}(c). \quad (2.6)$$

Given Z and c as above we define

$$\Phi_m(c, Z) = \sum_{\alpha \in \Gamma_m} c(\alpha) Z^\alpha \quad \text{and} \quad \Phi_m^o(c, Z) = \sum_{\alpha \in \Gamma_m^o} c(\alpha) Z^\alpha \quad (2.7)$$

Since c is symmetric we have $\Phi_m(c, Z) = m! \Phi_m^o(c, Z)$.

Remark 2.1 We notice that $\Phi_m(c, Z)$ is (in law) a multiple stochastic integral when the r.v.'s Z_k , $k \in \mathbb{N}$, are i.i.d. standard normal. In fact, W denoting a Brownian motion in \mathbb{R} , one has

$$\Phi_m^o(c, Z) = \int_0^\infty \int_0^{t_m} \cdots \int_0^{t_2} f(t_1, \dots, t_m) dW_{t_1} \cdots dW_{t_{m-1}} dW_{t_m}$$

when we take $Z_k = W_{k+1} - W_k$, $k \in \mathbb{N}$ and

$$f(t_1, \dots, t_m) = \sum_{\alpha \in \Gamma_m} c(\alpha) \prod_{i=1}^m \mathbf{1}_{[\alpha_i, \alpha_i+1)}(t_i).$$

We finally set

$$S(c, Z) = \sum_{m=1}^{\infty} \Phi_m(c, Z). \quad (2.8)$$

For finite sums, we will use the notation

$$S_N(c, Z) = \sum_{m=1}^N \Phi_m(c, Z). \quad (2.9)$$

For a random variable X we denote by $\kappa_4(X)$ the fourth cumulant, that is

$$\kappa_4(X) = \mathbb{E}(X^4) - 3\mathbb{E}(X^2)^2. \quad (2.10)$$

We will use the notation

$$\kappa_{4,m}(c) = \kappa_4(\Phi_m(c, \overline{Z})) \quad \text{when } \overline{Z}_1, \overline{Z}_2, \dots \text{ are i.i.d. } \sim \mathcal{N}(0, 1), \quad (2.11)$$

where, from now on, $\mathcal{N}(\mu, \sigma^2)$ denotes the normal law with mean μ and variance σ^2 . Recall that $\kappa_{4,1}(c) = 0$, because $\Phi_1(c, \overline{Z})$ is centered Gaussian. We also denote

$$\overline{\kappa}_{4,N}(c) = \sum_{m=1}^N \kappa_{4,m}^{1/4}(c). \quad (2.12)$$

Moreover it is known (see [18] or (B.8)) that

$$\delta_m(c) \leq \frac{1}{m!m|c|_m} \sqrt{\kappa_{4,m}(c)}, \quad m \geq 2. \quad (2.13)$$

In particular

$$\varepsilon_0(c, M) \leq \left(\sum_{m=1}^{\infty} \frac{M^{2m}}{m!m^2} \frac{\kappa_{4,m}(c)}{|c|_m^2} \right)^{1/2}. \quad (2.14)$$

3 Convergence of series for smooth test functions

The aim of this section is to discuss the convergence in law of the series of the form (2.8). The following estimates are immediate consequences of the isometry property and of Burkholder inequality (but see also next Lemma 5.1):

$$\|S(c, Z)\|_2 = N_0(c, 1) = \left(\sum_{m=0}^{\infty} m! |c|_m^2 \right)^{1/2} \quad \text{and} \quad (3.1)$$

$$\|S(c, Z)\|_p \leq N_0(c, \overline{M}_p^2(Z)) = \left(\sum_{m=0}^{\infty} \overline{M}_p^{2m}(Z) \times m! |c|_m^2 \right)^{1/2}. \quad (3.2)$$

Our first result consists in comparing $S(c, Z)$ and $S(c, \overline{Z})$ for two different sequences $Z_j, \overline{Z}_j, j \in \mathbb{N}$ of random variables.

Theorem 3.1 *Let $Z = (Z_k)_{k \in \mathbb{N}}$ and $\overline{Z} = (\overline{Z}_k)_{k \in \mathbb{N}}$ be two sequences of independent random variables such that $\mathbb{E}(Z_k) = \mathbb{E}(\overline{Z}_k) = 0$ and $\mathbb{E}(Z_k^2) = \mathbb{E}(\overline{Z}_k^2) = 1$. Recall (2.2), (2.4), (2.6) for the definition of $\overline{M}_p(Z)$ and $\overline{M}_p(\overline{Z})$, $N_0(c, M)$, $\varepsilon_0(c, M)$ respectively.*

A. *Let $M_3 = \overline{M}_3^2(Z) \vee \overline{M}_3^2(\overline{Z}) < \infty$. Then for every $f \in C_b^3(\mathbb{R})$,*

$$|\mathbb{E}(f(S(c, Z))) - \mathbb{E}(f(S(c, \overline{Z})))| \leq \frac{1}{3} \|f^{(3)}\|_{\infty} M_3^3 N_0(c, M_3)^2 \varepsilon_0(c, M_3). \quad (3.3)$$

B. *Suppose moreover that $\mathbb{E}(Z_k^3) = \mathbb{E}(\overline{Z}_k^3) = 0$ and $M_4 = \overline{M}_4^2(Z) \vee \overline{M}_4^2(\overline{Z}) < \infty$. Then for every $f \in C_b^4(\mathbb{R})$*

$$|\mathbb{E}(f(S(c, Z))) - \mathbb{E}(f(S(c, \overline{Z})))| \leq \frac{1}{12} \|f^{(4)}\|_{\infty} M_4^4 N_0(c, M_4)^2 \varepsilon_0^2(c, M_4). \quad (3.4)$$

Proof. **A.** Let $m, J \in \mathbb{N}$ and

$$S_{m,J}(c, Z) = \sum_{n=1}^m \sum_{\alpha \in \Gamma_n(J)} c(\alpha) Z^{\alpha}.$$

We will prove (3.3) with $S_{m,J}(c, Z)$ instead of $S(c, Z)$. Since the upper bound in the right hand side will not depend on m and J , the inequality for $S(c, Z)$ is obtained by passing to the limit with $m, J \rightarrow \infty$.

For $k = 1, \dots, J$ and $\theta \in [0, 1]$ we define the vector $\widehat{Z}^k(\theta)$ by

$$\widehat{Z}^k(\theta) = (Z_1, \dots, Z_{k-1}, \theta Z_k, \overline{Z}_{k+1}, \dots, \overline{Z}_J).$$

Then

$$\begin{aligned} & \mathbb{E}(f(S_{m,J}(c, Z))) - \mathbb{E}(f(S_{m,J}(c, \overline{Z}))) \\ &= \sum_{k=1}^J [\mathbb{E}(f(S_{m,J}(c, \widehat{Z}^k(1)))) - \mathbb{E}(f(S_{m,J}(c, \widehat{Z}^k(0))))] \\ & \quad - \sum_{k=1}^J [\mathbb{E}(f(S_{m,J}(c, \widehat{Z}^k(0)))) - \mathbb{E}(f(S_{m,J}(c, \widehat{Z}^{k-1}(1))))]. \end{aligned}$$

We write

$$S_{m,J}(c, \widehat{Z}^k(1)) = s_k + Z_k v_k$$

with

$$s_k = S_{m,J}(c, \widehat{Z}^k(0)) \quad \text{and} \quad v_k = c(k) + \sum_{n=2}^m \sum_{\substack{\beta \in \Gamma_{n-1}(J) \\ k \notin \beta}} c(k, \beta) (\widehat{Z}^k(0))^\beta$$

(recall that $(k, \beta) = (k, \beta_1, \dots, \beta_{m-1})$ for $\beta = (\beta_1, \dots, \beta_{m-1})$). Then, by using a development in Taylor series of order three,

$$\begin{aligned} f(S_{m,J}(c, \widehat{Z}^{k+1}(1))) - f(S_{m,J}(c, \widehat{Z}^{k+1}(0))) &= f'(s_k) Z_k v_k + \frac{1}{2} f''(s_k) Z_k^2 v_k^2 + \\ & \quad + \frac{1}{6} \int_0^1 f'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3 \theta d\theta. \end{aligned}$$

By taking expectation, by using independence and $\mathbb{E}(Z_k) = 0$, $\mathbb{E}(Z_k^2) = 1$, we obtain

$$\begin{aligned} \mathbb{E}(f(S_{m,J}(c, \widehat{Z}^k(1)))) - \mathbb{E}(f(S_{m,J}(c, \widehat{Z}^k(0)))) &= \frac{1}{2} \mathbb{E}(f''(s_k) v_k^2) + \\ & \quad + \frac{1}{6} \int_0^1 \mathbb{E}(f'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3) \theta d\theta. \end{aligned}$$

In a similar way we get

$$\begin{aligned} \mathbb{E}(f(S_{m,J}(c, \widehat{Z}^k(0)))) - \mathbb{E}(f(S_{m,J}(c, \widehat{Z}^{k-1}(1)))) &= -\frac{1}{2} \mathbb{E}(f''(s_k) v_k^2) + \\ & \quad - \frac{1}{6} \int_0^1 \mathbb{E}(f'''(s_k + \theta \overline{Z}_k v_k) \overline{Z}_k^3 v_k^3) \theta d\theta. \end{aligned}$$

The term containing f'' is the same in the two cases, so it cancels when taking sums. Then we obtain

$$\begin{aligned} \mathbb{E}(f(S_{m,J}(c, Z))) - \mathbb{E}(f(S_{m,J}(c, \overline{Z}))) &= \frac{1}{6} \sum_{k=1}^J \int_0^1 \mathbb{E}(f'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3) \theta d\theta \\ & \quad - \frac{1}{6} \sum_{k=1}^J \int_0^1 \mathbb{E}(f'''(s_k + \theta \overline{Z}_k v_k) \overline{Z}_k^3 v_k^3) \theta d\theta. \end{aligned}$$

Since Z_k is independent of v_k we have

$$|\mathbb{E}(f'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3)| \leq \|f'''\|_\infty \mathbb{E}(|Z_k|^3) \mathbb{E}(|v_k|^3) \leq \|f'''\|_\infty M_3^3(Z) \mathbb{E}(|v_k|^3).$$

Now, using (3.2) with $p = 3$, we obtain (recall that $M_3 = \overline{M}_3^2(Z) \vee \overline{M}_3^2(\overline{Z})$)

$$\mathbb{E}(|v_k|^3) \leq \left(\sum_{n=0}^{m-1} M_3^n \times n! \sum_{\substack{|\beta|=n \\ k \notin \beta}} c^2(k, \beta) \right)^{3/2}$$

and so

$$|\mathbb{E}(f'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3)| \leq \|f'''\|_\infty M_3^3 \left(\sum_{n=0}^{m-1} M_3^n \times n! \sum_{\substack{|\beta|=n \\ k \notin \beta}} c^2(k, \beta) \right)^{3/2}.$$

The same estimate holds if we take \overline{Z}_k instead of Z_k . We conclude that

$$\begin{aligned} |\mathbb{E}(f(S_{m,J}(c, Z))) - \mathbb{E}(f(S_{m,J}(c, \overline{Z})))| &\leq \frac{1}{3} \|f'''\|_\infty M_3^3 \sum_{k=1}^J \left(\sum_{n=0}^{m-1} M_3^n \times n! \sum_{|\beta|=n} c^2(k, \beta) \right)^{3/2} \\ &\leq \frac{1}{3} \|f'''\|_\infty M_3^3 \max_{k \leq J} \left(\sum_{n=0}^{m-1} M_3^n \times n! \sum_{|\beta|=n} c^2(k, \beta) \right)^{1/2} \times \\ &\quad \times \sum_{k=1}^J \sum_{n=0}^{m-1} M_3^n \times n! \sum_{|\beta|=n} c^2(k, \beta). \end{aligned}$$

Notice that

$$\sum_{k=1}^J \sum_{n=0}^{m-1} M_3^n \times n! \sum_{|\beta|=n} c^2(k, \beta) = \sum_{n=0}^{m-1} M_3^n \times n! \sum_{|\alpha|=n+1} c^2(\alpha) \leq N_0(c, M_3)^2$$

and

$$\max_{k \leq J} \left(\sum_{n=0}^{m-1} M_3^n \times n! \sum_{|\beta|=n} c^2(k, \beta) \right)^{1/2} \leq \varepsilon_0(c, M).$$

We conclude that

$$|\mathbb{E}(f(S_{m,J}(c, Z))) - \mathbb{E}(f(S_{m,J}(c, \overline{Z})))| \leq \frac{1}{3} M_3^3 \|f'''\|_\infty \varepsilon_0(c, M_3) N_0(c, M_3)^2$$

so the proof of (3.3) is completed. The proof of (3.4) is identical: one just go further to order 4 in the Taylor expansion. \square

We discuss now the convergence to a Gaussian random variable. This immediately follows from the previous result and from Theorem 3 in [9]. We consider a sequence $c^{(n)} = (c^{(n)}(\alpha))_{\alpha \in \Gamma}, n \in \mathbb{N}$ of coefficients and the corresponding stochastic series $S(c^{(n)}, Z)$. Our assumptions will be the following:

$$\begin{aligned} i) \quad & \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k \geq N} k! \sum_{|\alpha|=k} |c^{(n)}(\alpha)|^2 = 0, \\ ii) \quad & \lim_{n \rightarrow \infty} k! \sum_{|\alpha|=k} |c^{(n)}(\alpha)|^2 =: \sigma_k^2, \\ iii) \quad & \sum_{k=1}^{\infty} \sigma_k^2 = \sigma^2 \\ iv) \quad & \lim_{n \rightarrow \infty} \kappa_{4,k}(c^{(n)}) = 0, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{3.5}$$

Theorem 3.2 *Let $Z = (Z_k)_{k \in \mathbb{N}}$ be a sequence of independent random variables such that $\mathbb{E}(Z_k) = 0$, $\mathbb{E}(Z_k^2) = 1$ and $M_3 = \max_{k \in \mathbb{N}} \mathbb{E}(|Z_k|^3) < \infty$. Consider also a sequence $c^{(n)} = (c^{(n)}(\alpha))_{\alpha \in \Gamma}, n \in \mathbb{N}$ of coefficients which satisfy (3.5). Then $\lim_{n \rightarrow \infty} S(c^{(n)}, Z) = \mathcal{N}(0, \sigma^2)$ in law.*

Proof. Notice first that the sequence of laws of $S(c^{(n)}, Z)$ is tight, so the only thing to be proven is that any limit point is in fact $\mathcal{N}(0, \sigma^2)$. Let $\overline{Z}_k, k \in \mathbb{N}$ be a sequence of independent and standard normal random variables, so that $S(c^{(n)}, \overline{Z})$ is an infinite sum of multiple stochastic integrals (see Remark 2.1). Then, under the hypothesis (3.5), Hu and Nualart proved (see Theorem 3 in [9]) that $\lim_{n \rightarrow \infty} S(c^{(n)}, \overline{Z}) = \mathcal{N}(0, \sigma^2)$ in law. And (3.3) guarantees that $\lim_{n \rightarrow \infty} S(c^{(n)}, Z)$ is the same. \square

4 Variational calculus using a splitting method

In order to study the convergence in total variation and some related invariance principles, our specific point is to consider a class of random variables which have a regularity property allowing one to extrapolate an “absolutely continuous noise”.

4.1 The splitting procedure

We say that the law of the random variable $Z \in \mathbb{R}$ is locally lower bounded by the Lebesgue measure if there exists $z \in \mathbb{R}$ and $\varepsilon, r > 0$ such that for every non negative and measurable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$

$$A(z, r, \varepsilon) : \quad \mathbb{E}(f(Z)) \geq \varepsilon \int f(\xi - z) 1_{B(0, r)}(\xi - z) d\xi. \quad (4.1)$$

We denote by $\mathcal{L}(z, r, \varepsilon)$ the class of the random variables which verify $A(z, r, \varepsilon)$. Given $r > 0$ we consider the functions $\theta_r, \psi_r : \mathbb{R} \rightarrow \mathbb{R}_+$ defined by

$$\theta_r(t) = 1 - \frac{1}{1 - (\frac{t}{r} - 1)^2} \quad \psi_r(t) = 1_{\{|t| \leq r\}} + 1_{\{r < |t| \leq 2r\}} e^{\theta_r(|t|)}. \quad (4.2)$$

If $Z \in \mathcal{L}(z, r, \varepsilon)$ then

$$\mathbb{E}(f(Z)) \geq \varepsilon \int f(\xi - z) \psi_r(|\xi - z|^2) d\xi. \quad (4.3)$$

The advantage of $\psi_r(|\xi - z|^2)$ is that it is a smooth function (which replaces the indicator function of the ball) and (it is easy to check) that for each $l \in \mathbb{N}, p \geq 1$ there exists a constant $C_{l,p} \geq 1$ such that

$$\sup_{t \in \mathbb{R}} \psi_r(t) |\theta_r^{(l)}(|t|)|^p \leq \frac{C_{l,p}}{r^{lp}} \quad (4.4)$$

where $\theta_r^{(l)}$ denotes the derivative of order l of θ_r . Moreover, in Proposition 3.1 in [1] it is proved that if $Z \in \mathcal{L}(z, r, \varepsilon)$ then Z admits the following decomposition (the equality is understood as identity of laws):

$$Z = \chi V + (1 - \chi)U \quad (4.5)$$

where χ, U, V are independent random variables with the following laws:

$$\begin{aligned} \mathbb{P}(\chi = 1) &= \varepsilon m(r) \quad \text{and} \quad \mathbb{P}(\chi = 0) = 1 - \varepsilon m(r), \\ \mathbb{P}(V \in d\xi) &= \frac{1}{m(r)} \psi_r(|\xi - z|)^2 d\xi \\ \mathbb{P}(U \in d\xi) &= \frac{1}{1 - \varepsilon m(r)} (\mathbb{P}(Z \in d\xi) - \varepsilon \psi_r(|\xi - z|^2) d\xi) \end{aligned} \quad (4.6)$$

with

$$m(r) = \int \psi_r(|\xi|^2) d\xi. \quad (4.7)$$

Assumption 4.1 *From now on, we consider functionals of a sequence of independent random variables $Z_k \in \mathbb{R}, k \in \mathbb{N}$, having all moments and such that $Z_k \in \mathcal{L}(z_k, r, \varepsilon)$ for every $k \in \mathbb{N}$. Remark that Z_k are not identically distributed but we assume that r and ε are the same for all of them (on the contrary, z_k may depend on k).*

4.2 Differential operators and Sobolev spaces

We use the stochastic differential calculus (an abstract finite dimensional Malliavin type calculus) based on $V_k, k \in \mathbb{N}$ settled in [4] [2] and, for this kind of splitting, in [1]. The crucial point is that the law of V_k is absolutely continuous and has the nice density $\psi_r(|z - z_k|^2)$. We recall the results we need in the following sections.

We denote by \mathcal{P} the subspace of the measurable functions $\Phi : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ that are polynomials. So $\Phi \in \mathcal{P}$ with degree n means that there exists $n \in \mathbb{N}$ and $(c(\beta))_{\beta \in \cup_{m=1}^n \Gamma_m}$ such that

$$\Phi(z) = \sum_{n=1}^m \sum_{\beta \in \Gamma_n} c(\beta) z^\beta. \quad (4.8)$$

We define the space of simple functionals

$$\mathcal{S} = \{F = \Phi(Z) : \Phi \in \mathcal{P}\} \quad (4.9)$$

where \mathcal{P} is the space of the polynomials defined above. For $F = \Phi(Z) \in \mathcal{S}$ we define the derivative operator

$$D_k F = \frac{\partial}{\partial V_k} F = \chi_k \frac{\partial}{\partial Z_k} F = \chi_k \partial_k \Phi(Z). \quad (4.10)$$

We look to $DF = (D_k F)_{k \in \mathbb{N}}$ as to a random element of the Hilbert space

$$\mathcal{H} = \left\{ z \in \mathbb{R}^{\mathbb{N}} : |z|_{\mathcal{H}}^2 := \sum_{k=1}^{\infty} z_k^2 < \infty \right\}. \quad (4.11)$$

Moreover we define the higher order derivatives in the following way. Let $n \in \mathbb{N}$ be fixed and let $\alpha = (\alpha_1, \dots, \alpha_n)$. For $F = \Phi(Z) \in \mathcal{S}$, we define

$$D_\alpha^{(n)} F = D_{\alpha_n} \cdots D_{\alpha_1} F = \left(\prod_{j=1}^n \chi_{\alpha_j} \right) (\partial_{\alpha_n} \cdots \partial_{\alpha_1} \Phi)(Z) = \left(\prod_{j=1}^n \chi_{\alpha_j} \right) \partial_\alpha \Phi(Z). \quad (4.12)$$

We look to $D^{(n)} F = (D_\alpha^{(n)} F)_{\alpha \in \Gamma_n}$ as to a random element of $\mathcal{H}^{\otimes n}$. For $n = 1$, we write $D^{(1)} F = DF$.

We define now

$$LF = - \sum_{k=1}^{\infty} (D_k D_k F + D_k F \times \Theta_k) \quad \text{with} \quad \Theta_k = 2\theta'_r(|V_k - z_k|^2)(V_k - z_k). \quad (4.13)$$

Elementary integration by parts gives the following duality relation: for every $F, G \in \mathcal{P}$

$$\mathbb{E}(\langle DF, DG \rangle_{\mathcal{H}}) = \mathbb{E}(FLG) = \mathbb{E}(GLF). \quad (4.14)$$

We define now the Sobolev norms. For $q \geq 1$ we set

$$|F|_{1,q} = \sum_{n=1}^q |D^{(n)} F|_{\mathcal{H}^{\otimes n}} \quad \text{and} \quad |F|_q = |F| + |F|_{1,q}. \quad (4.15)$$

Moreover we define

$$\|F\|_{1,q,p} = (\mathbb{E}(|F|_{1,q}^p))^{1/p}, \quad \|F\|_{q,p} = (\mathbb{E}(|F|_q^p))^{1/p} \quad (4.16)$$

and

$$\|F\|_{1,q,p} = \|F\|_{1,q,p} + \|LF\|_{q-2,p}, \quad \|F\|_{q,p} = \|F\|_p + \|F\|_{1,q,p}. \quad (4.17)$$

Finally we define the Sobolev spaces

$$\mathbb{D}^{q,p} = \overline{\mathcal{S}}^{\|\cdot\|_{q,p}}, \quad \mathbb{D}^{q,\infty} = \cap_{p=1}^{\infty} \mathbb{D}^{q,p} \quad \mathbb{D}^{\infty} = \cap_{q=1}^{\infty} \mathbb{D}^{q,\infty}. \quad (4.18)$$

Notice that the duality relation (4.14) implies that the operators $D^{(n)}$ and L are closable so we may extend these operators to $\mathbb{D}^{q,p}$ in a standard way.

4.3 Integration by parts formula

In this section we recall some results from [2] and [4]. All these results are stated in that papers for a functional F which depends on Z_1, \dots, Z_J only (a finite number of random variables). But all of them extend in a trivial way to $F \in \mathbb{D}^{q,p}$.

We recall first the basic computational rules and the integration by parts formula. For $\phi \in C^1(\mathbb{R}^M) \in (\mathbb{D}^{2,\infty})^M$ we have

$$D\phi(F) = \sum_{j=1}^M \partial_j \phi(F) DF^j, \quad (4.19)$$

and for $\phi \in C^2(\mathbb{R}^M)$

$$L\phi(F) = \sum_{j=1}^M \partial_j \phi(F) LF^j - \frac{1}{2} \sum_{i,j=1}^M \partial_i \partial_j \phi(F) \langle DF^i, DF^j \rangle_{\mathcal{H}}. \quad (4.20)$$

In particular for $F, G \in \mathbb{D}^{2,\infty}$

$$L(FG) = FLG + GLF - \langle DF, DG \rangle_{\mathcal{H}}. \quad (4.21)$$

For a functional $F = (F^1, \dots, F^M) \in (\mathbb{D}^{2,\infty})^M$ we define the Malliavin covariance matrix σ_F by

$$\sigma_F^{i,j} = \langle DF^i, DF^j \rangle_{\mathcal{H}}, \quad i, j = 1, \dots, M. \quad (4.22)$$

The lower eigenvalue of σ_F is

$$\lambda_F = \inf_{|\xi|=1} \langle \sigma_F \xi, \xi \rangle = \inf_{|\xi|=1} \sum_{k=1}^{\infty} \langle D_{(k)} F, \xi \rangle_{\mathcal{H}}^2. \quad (4.23)$$

If σ_F is invertible we denote $\gamma_F = \sigma_F^{-1}$. Moreover we denote

$$\overline{\sigma}_F(p) = 1 \vee \mathbb{E}((\det \sigma_F)^{-p}), \quad \overline{\lambda}_F(p) = 1 \vee \mathbb{E}((\lambda_F)^{-p}) \quad (4.24)$$

We are now able to give the Malliavin integration by parts formulae. Here, $C_p^\infty(\mathbb{R}^M)$ denotes the set of the infinitely differentiable functions whose derivatives, of any order, have polynomial growth.

Theorem 4.2 *Let $F = (F^1, \dots, F^M)$, $F_i \in \mathbb{D}^{2,\infty}$ and $G \in \mathbb{D}^{1,\infty}$ be such that $\overline{\sigma}_F(p) < \infty$ for every $p \geq 1$. Then for every $\phi \in C_p^\infty(\mathbb{R}^M)$ and every $i = 1, \dots, M$*

$$\mathbb{E}(\partial_i \phi(F) G) = \mathbb{E}(\phi(F) H_i(F, G)) \quad (4.25)$$

with

$$H_i(F, G) = G \gamma_F LF + \langle D(G \gamma_F), DF \rangle_{\mathcal{H}} \quad (4.26)$$

Moreover let $m \in \mathbb{N}$, $m \geq 2$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \{1, \dots, M\}^m$. Suppose that $F = (F^1, \dots, F^M)$, $F_i \in \mathbb{D}^{m+1,\infty}$ and $G \in \mathbb{D}^{m,\infty}$. Then

$$\mathbb{E}(\partial_\alpha \phi(F) G) = \mathbb{E}(\phi(F) H_\alpha(F, G)) \quad (4.27)$$

with $H_\alpha(F, G)$ defined by $H_{(\alpha_1, \dots, \alpha_m)}(F, G) := H_{\alpha_m}(F, H_{(\alpha_1, \dots, \alpha_{m-1})}(F, G))$.

Proof. We give here only a sketch of the proof, a detailed one can be found e.g. in [2] and [4]. Using the chain rule $D\phi(F) = \nabla \phi(F) DF$ so that

$$\langle D\phi(F), DF \rangle_{\mathcal{H}} = \nabla \phi(F) \langle DF, DF \rangle_{\mathcal{H}} = \nabla \phi(F) \sigma_F.$$

It follows that $\nabla\phi(F) = \gamma_F \langle D\phi(F), DF \rangle_{\mathcal{H}}$. Then, by using (4.21) and the duality formula (4.14),

$$\begin{aligned} \mathbb{E}(G\nabla\phi(F)) &= \mathbb{E}(G\gamma_F \langle D\phi(F), DF \rangle_{\mathcal{H}}) = \mathbb{E}(G\gamma_F(L(\phi(F)F) - \phi(F)LF + FL\phi(F))) \\ &= \mathbb{E}(\phi(F)(FL(G\gamma_F) + G\gamma_F LF + L(G\gamma_F F))). \end{aligned}$$

We use once again (4.21) in order to obtain $H_i(F, G)$ in (4.26). By iteration one obtains the higher order integration by parts formulae. \square

We give now useful estimates for the weights which appear in (4.27):

Lemma 4.3 *Let $F \in \mathcal{S}^M$ be such that $\bar{\sigma}_F(p) < \infty$ for every $p \geq 1$ and let $G \in \mathcal{S}$. Then for each $m, q \in \mathbb{N}$ there exists a universal constant $C \geq 1$ (depending on M, m, q only) such that for every multi-index α with $|\alpha| \leq q$ one has*

$$|H_\alpha(F, G)|_m \leq C(1 \vee (\det \sigma_F)^{-1})^{q(m+1)}(1 + |F|_{1, m+q+2}^{2Mq(m+2)} + |LF|_{m+q}^q) |G|_{m+q}. \quad (4.28)$$

In particular we have

$$\|H_\alpha(F, G)\|_p \leq C\bar{\sigma}_F(2pq)(1 + \|F\|_{1, q+2, 4p}^{6qM}) \|G\|_{q, 4p} \quad (4.29)$$

The proof is long so we skip it, details may be found in [4] and in [2] Theorem 3.4. We will also need the following:

Lemma 4.4 *For every $l \in \mathbb{N}$ there exists a constant $C_l \geq 1$ such that for every $q \in \mathbb{N}$, $p \geq 1$ and $G \in \mathbb{D}^{q, \infty}$,*

$$\|G^l\|_{q, p} \leq C_l \|G\|_{q, 2^l p}^l \quad (4.30)$$

The proof is straightforward so we skip it.

4.4 Regularization and non degeneracy

In this section we consider a functional $F \in (\mathbb{D}^{2, \infty})^M$. As it is clear from (4.28), a delicate point in using the integration by parts formulae is to ensure that the functionals at hand are non degenerate, that is $\det \sigma_F > 0$. And in fact this is never true almost surely: this is because $\chi_1 = \dots = \chi_m = 0$ with strictly positive probability. In order to bypass this difficulty we use a regularization argument involving the lowest eigenvalue λ_F of σ_F . We denote

$$\lambda_{\delta, \eta, q}(F) = \delta^{-1} \mathbb{P}(\lambda_F \leq \eta)^{1/q} + \eta^{-1}, \quad \delta, \eta > 0, q \in \mathbb{N}. \quad (4.31)$$

We also set

$$\gamma_\delta(z) = \frac{1}{m(1)\sqrt{\delta}} \psi_1(\delta^{-1} |z|^2) \quad (4.32)$$

where ψ_1 is the function defined in (4.2) and $m(1)$ is the normalization constant from (4.7) (with $r = 1$). For $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we denote

$$f_\delta = f * \gamma_\delta, \quad (4.33)$$

the symbol $*$ denoting convolution. We also consider a supplementary random variable

$$Z_0 \sim \frac{1}{m(1)} \psi_1(|z|^2) dz$$

which we assume to be independent of $Z_k, k \in \mathbb{N}$, and we define

$$F_\delta = F + \sqrt{\delta} Z_0. \quad (4.34)$$

Lemma 4.5 *Let $q \in \mathbb{N}$, $M \in \mathbb{N}$ with $M \geq 1$ and $p_1, p_2, p_3 > 0$ be such that $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$. Then there exists a constant C such that for every $\delta, \eta > 0$, every multi-index α with $|\alpha| = q$, every measurable function $f : \mathbb{R}^M \rightarrow \mathbb{R}$ and every $F \in (\mathbb{D}^{q+2, \infty})^M$ and $G \in \mathbb{D}^{q+1, \infty}$ one has*

$$|\mathbb{E}(\partial_\alpha f_\delta(F)G)| \leq C \lambda_{\delta, \eta, Mqp_1}^{Mqp_1}(F) \|f\|_\infty (1 + \|F\|_{1, q+2, 4Mqp_2}^{4Mq}) \|G\|_{q, qp_3}. \quad (4.35)$$

Proof. We notice first that

$$\mathbb{E}(\partial_\alpha f_\delta(F)G) = \mathbb{E}(\partial_\alpha f(F_\delta)G).$$

We work with the integration by parts formula based on Z_0 and on $Z_k, k \in \mathbb{N}$. Then

$$D_k F_\delta = D_k F \text{ for } k \geq 1 \text{ and } D_0 F_\delta = \sqrt{\delta} \text{ for } k = 0.$$

and $D^{(n)} F_\delta = D^{(n)} F$ for $n \geq 2$. So $\|F_\delta\|_{1, q+2, p} \leq \|F\|_{1, q+2, p} + \sqrt{\delta}$. Moreover

$$L F_\delta = L F + \sqrt{\delta} (\ln \psi_1)'(|Z_0|^2) \times 2Z_0$$

so that, by (4.4) we obtain $\|L F_\delta\|_{q+2, p} \leq \|L F\|_{q+2, p} + C\sqrt{\delta}$. We conclude that

$$\|F_\delta\|_{1, q+2, p} \leq \|F\|_{1, q+2, p} + C\sqrt{\delta}.$$

We look now to the covariance matrix:

$$\begin{aligned} \langle \sigma_{F_\delta}, \xi, \xi \rangle &= \sum_{k=0}^{\infty} \langle D_k F_\delta, \xi \rangle^2 = \langle D_0(\sqrt{\delta} Z_0), \xi \rangle^2 + \sum_{k=0}^{\infty} \langle D_k F, \xi \rangle^2 \\ &\geq \delta |\xi|^2 + \lambda_F |\xi|^2, \end{aligned}$$

so the lowest eigenvalue of σ_{F_δ} verifies $\lambda_{F_\delta} \geq \delta + \lambda_F$. Using the integration by parts formulae (4.27) for F_δ we obtain

$$\mathbb{E}(\partial_\alpha f(F_\delta)G) = \mathbb{E}(f(F_\delta)H_\alpha(F_\delta, G)).$$

By (4.28) with $m = 0$

$$|H_\alpha(F_\delta, G)| \leq C(1 \vee (\lambda_{F_\delta})^{-1})^{Mq} (1 + |F|_{1, q+2}^{4Mq} + |L F|_q^q) |G|_q$$

so that, using Hölder's inequality

$$\mathbb{E}(|H_\alpha(F_\delta, G)|) \leq C(1 \vee \mathbb{E}((\lambda_{F_\delta})^{-Mqp_1})^{1/p_1} (1 + \|F\|_{1, q+2, Mqp_2}^{4Mq} + \|L F\|_{q, qp_2}^{4Mq}) \|G\|_{q, qp_3}.$$

We write now

$$\begin{aligned} \mathbb{E}((\lambda_{F_\delta})^{-Mqp_1}) &= \mathbb{E}((\lambda_{F_\delta})^{-Mqp_1} 1_{\{\lambda_F \leq \eta\}}) + \mathbb{E}((\lambda_{F_\delta})^{-Mqp_1} 1_{\{\lambda_F > \eta\}}) \\ &\leq \delta^{-Mqp_1} \mathbb{P}(\lambda_F \leq \eta) + \eta^{-Mqp_1} \\ &\leq (\delta^{-1} \mathbb{P}(\lambda_F \leq \eta))^{1/Mqp_1} + \eta^{-1} = \lambda_{\delta, \eta, Mqp_1}^{Mqp_1}(F). \end{aligned}$$

We conclude that

$$\begin{aligned} \mathbb{E}(\partial_\alpha f_\delta(F)G) &= |\mathbb{E}(\partial_\alpha f(F_\delta)G)| \leq \|f\|_\infty \mathbb{E}(|H_\alpha(F_\delta, G)|) \\ &\leq C \lambda_{\delta, \eta, Mqp_1}^{Mq}(F) \|f\|_\infty (1 + \|F\|_{1, q+2, Mqp_2}^{4Mq} + \|L F\|_{q, 4qp_2}^{4Mq}) \|G\|_{q, qp_3} \end{aligned}$$

□

In order to pass from f_δ to f we will use the following lemma:

Lemma 4.6 *Let $M \in \mathbb{N}$, $M \geq 1$. There exist constants $C, p, a \geq 1$ such that for every $\eta > 0, \delta > 0$, every $F \in (\mathbb{D}^{3, p})^M$ and every bounded and measurable $f : \mathbb{R}^M \rightarrow \mathbb{R}$ one has*

$$|\mathbb{E}(f(F)) - \mathbb{E}(f_\delta(F))| \leq C \|f\|_\infty \left(\mathbb{P}(\lambda_F < \eta) + \frac{\sqrt{\delta}}{\eta^p} (1 + \|F\|_{3, p})^a \right) \quad (4.36)$$

The above Lemma is Lemma 2.5 in [2]. There γ_δ is the Gaussian density of covariance δ but the proof is exactly the same with γ_δ defined in (4.32).

5 Sobolev norms and non-degeneracy for stochastic series

The stochastic series $S(c, Z) = \sum_{m=1}^{\infty} \Phi_m(c, Z)$ are a natural generalization of the decomposition in Wiener chaoses - indeed, if Z_k are standard normal then the $\Phi_m(c, Z)$'s represent multiple stochastic integrals of order m (Remark 2.1). The aim of this section is to obtain estimates of the Sobolev norms of $S(c, Z)$ and of $LS(c, Z)$ which are analogous to the ones we have in the Gaussian case. To this purpose, it is useful to introduce random variables taking values on a Hilbert space \mathcal{U} that are derivable in Malliavin sense. In fact, $DS(c, Z)$ is a r.v. in \mathcal{H} (see (4.11)) and can be written again as a stochastic series whose coefficients are in \mathcal{H} . So, in order to handle properly our problem, we consider stochastic series whose coefficients $c(\alpha)$ belongs to a separable Hilbert space \mathcal{U} . We denote with $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ and $|\cdot|_{\mathcal{U}}$ the associated inner product and norm, respectively. We set $L_{\mathcal{U}}^p = \{F \in \mathcal{U} : \|F\|_{\mathcal{U},p} := (\mathbb{E}(|F|_{\mathcal{U}}^p))^{1/p} < \infty\}$. Even if $F \in \mathbb{R}$ (as it is the case in our paper), the derivative DF takes values in $\mathcal{U} = \mathcal{H}$ which is a Hilbert space.

We set

$$\mathcal{H}(\mathcal{U}) = \{x \in \mathcal{U}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|_{\mathcal{U}}^2 < \infty\}. \quad (5.1)$$

$\mathcal{H}(\mathcal{U})$ is clearly a Hilbert space, the inner product and the norm being given by

$$\langle x, y \rangle_{\mathcal{H}(\mathcal{U})} = \sum_{k=1}^{\infty} \langle x_k, y_k \rangle_{\mathcal{U}} \quad \text{and} \quad |x|_{\mathcal{H}(\mathcal{U})} = \left(\sum_{k=1}^{\infty} |x_k|_{\mathcal{U}}^2 \right)^{1/2}$$

respectively. Notice that $\mathcal{H}(\mathbb{R})$ is the space \mathcal{H} defined in (4.11). Remark also that $\mathcal{H}(\mathcal{H}(\mathcal{U})) = \mathcal{H}(\mathcal{U})^{\otimes 2}$, and more generally $\mathcal{H}(\mathcal{H}(\mathcal{U})^{\otimes n}) = \mathcal{H}(\mathcal{U})^{\otimes(n+1)}$.

Let A be a random variable taking values in \mathcal{U} which is measurable with respect to $\sigma(Z_i, i = 1, 2, \dots)$ and take $p \geq 1$. We set $\mathbb{D}_{\mathcal{U}}^{0,p} = L_{\mathcal{U}}^p$ and we set

$$\|A\|_{\mathcal{U},0,p} \equiv \|A\|_{\mathcal{U},p} = \| |A|_{\mathcal{U}} \|_p.$$

We say that $A \in \mathbb{D}_{\mathcal{U}}^{1,p}$ if $A \in L_{\mathcal{U}}^p$, $\langle A, h \rangle_{\mathcal{U}} \in \mathbb{D}^{1,p}$ for every $h \in \mathcal{U}$ and there exists $DA \in \mathcal{H}(\mathcal{U})$ such that

$$DA \in L_{\mathcal{H}(\mathcal{U})}^p \quad \text{and} \quad \langle (DA)_k, h \rangle_{\mathcal{U}} = D_k \langle A, h \rangle_{\mathcal{U}}, \quad k \in \mathbb{N}.$$

In the following, we use the notation $D_k A = (DA)_k$, $k \in \mathbb{N}$. For $A \in \mathbb{D}_{\mathcal{U}}^{1,p}$ we define the Sobolev norm

$$\|A\|_{\mathcal{U},1,p} = \|A\|_{\mathcal{U},p} + \|DA\|_{\mathcal{H}(\mathcal{U}),p}.$$

Note that if $\mathcal{U} = \mathbb{R}$ then the above definition $F \in \mathbb{D}^{1,p}$ agrees with the standard definition: $\mathbb{D}_{\mathbb{R}}^{1,p} \equiv \mathbb{D}^{1,p}$.

This reasoning can be iterated in order to define a random variable A in \mathcal{U} which is $q \geq 2$ times differentiable: for $p \geq 1$, we say that $A \in \mathbb{D}_{\mathcal{U}}^{q,p}$ if $A \in \mathbb{D}_{\mathcal{U}}^{q-1,p}$, $\langle A, h \rangle_{\mathcal{U}} \in \mathbb{D}^{q,p}$ for every $h \in \mathcal{U}$ and there exists $D^{(q)}A \in \mathcal{H}(\mathcal{U})^{\otimes q}$ such that

$$D^{(q)}A \in L_{\mathcal{H}(\mathcal{U})^{\otimes q}}^p \quad \text{and} \quad \langle (D^{(q)}A)_{\alpha}, h \rangle_{\mathcal{U}} = D_{\alpha}^{(q)} \langle A, h \rangle_{\mathcal{U}}, \quad |\alpha| = q.$$

We set $D_{\alpha}^{(q)}A = (D^{(q)}A)_{\alpha}$, $|\alpha| = q$, and

$$\|A\|_{\mathcal{U},q,p} = \|A\|_{\mathcal{U},q-1,p} + \|D^{(q)}A\|_{\mathcal{H}(\mathcal{U})^{\otimes q},p} = \|A\|_{\mathcal{U},p} + \sum_{j=1}^q \|D^{(j)}A\|_{\mathcal{H}(\mathcal{U})^{\otimes j},p}.$$

As an example, take F a random variable in \mathbb{R} . Then $F \in \mathbb{D}^{2,p}$ if $F \in \mathbb{D}^{1,p}$ (standard definition) and $DF \in \mathbb{D}_{\mathcal{H}}^{1,p}$ following the above definition, that is looking at $A = DF$ as a random variable taking values in the Hilbert space $\mathcal{U} = \mathcal{H}$, and one has $\|F\|_{2,p} = \|F\|_{\mathcal{U},2,p}$ with $\mathcal{U} = \mathbb{R}$. And in general, $F \in \mathbb{D}^{q,p}$ if for every $1 \leq k \leq q-1$, $D^{(k)}F \in \mathbb{D}_{\mathcal{H}^{\otimes k}}^{1,p}$ and one has $\|F\|_{q,p} = \|F\|_{\mathcal{U},q,p}$ with $\mathcal{U} = \mathbb{R}$.

We consider now the random variable $S(c, Z) = \sum_{\alpha} c(\alpha) Z^{\alpha}$ in (2.8) with $c(\alpha) \in \mathcal{U}$ for every α . We denote

$$|c|_{\mathcal{U},m} = \left(\sum_{\alpha \in \Gamma_m} |c(\alpha)|_{\mathcal{U}}^2 \right)^{1/2}. \quad (5.2)$$

For $q, p \in \mathbb{N}$ and $M \in \mathbb{R}$ we set

$$N_q(c, M) = \left(\sum_{m=q}^{\infty} M^{m-q} \times \frac{m!}{(m-q)!} \times m! |c|_{\mathcal{U},m}^2 \right)^{1/2} \quad (5.3)$$

(for a comparison, see (2.4) for $\mathcal{U} = \mathbb{R}$).

We recall that we are assuming that $Z_k, k \in \mathbb{N}$ are independent and

$$\mathbb{E}(Z_k) = 0 \quad \text{and} \quad \mathbb{E}(Z_k^2) = 1. \quad (5.4)$$

The following basic relations are immediate consequences of the isometry property for Hilbert space valued discrete martingales. Let $\Phi_m(c, Z) = \sum_{|\alpha|=m} c(\alpha) Z^{\alpha}$ with $c(\alpha) \in \mathcal{U}$. Then

$$\mathbb{E}(\langle \Phi_m(c, Z), \Phi_{m'}(c, Z) \rangle_{\mathcal{U}}) = \begin{cases} 0 & m \neq m' \\ m! |c|_{\mathcal{U},m}^2 & m = m'. \end{cases} \quad (5.5)$$

Lemma 5.1 *Let $Z_k, k \in \mathbb{N}$, be independent such that (5.4) holds. Let $S(c, Z) = \sum_{m=1}^{\infty} \Phi_m(Z)$ and $N_q(c, M)$ be defined in (5.3).*

(i) *The series $S(c, Z)$ is convergent in $L_{\mathcal{U}}^2$ if and only if $N_0(c, 1) < \infty$ and in this case*

$$\|S(c, Z)\|_{\mathcal{U},2} = N_0(c, 1) = \left(\sum_{m=1}^{\infty} m! |c|_{\mathcal{U},m}^2 \right)^{1/2}.$$

(ii) *Let b_p be the constant in the Burkholder's inequality (see (A.1)) and*

$$M_p = \sqrt{2} b_p M_p(Z), \quad \text{with} \quad M_p(Z) = \sup_{k \in \mathbb{N}} \|Z_k\|_p. \quad (5.6)$$

Then

$$\|S(c, Z)\|_{\mathcal{U},p} \leq N_0(c, M_p^2) = \left(\sum_{m=1}^{\infty} M_p^{2m} m! |c|_{\mathcal{U},m}^2 \right)^{1/2}. \quad (5.7)$$

Remark 5.2 *As an immediate consequence one has the following estimate for multiple integrals in Wiener chaoses: $\|I_m(f)\|_p \leq \sqrt{2} b_p M_p(G) \sqrt{m!} \|f\|_{L^2(\mathbb{R}_+^m)}$ where $G \sim \mathcal{N}(0, 1)$. More accurate estimates concerning the Gaussian chaoses can be found in Latala [12].*

Proof of Lemma 5.1. (i) immediately follows from (5.5). As for (ii), we fix $J \in \mathbb{N}$ and we denote $|c|_{\mathcal{U},m,J}^2 = \sum_{\alpha \in \Gamma_m(J)} |c(\alpha)|_{\mathcal{U}}^2$ and

$$\Phi_{m,J}(c, Z) = \sum_{\alpha \in \Gamma_m(J)} c_{\alpha} Z^{\alpha}, \quad \Phi_{m,J}^o(c, Z) = \sum_{\alpha \in \Gamma_m^o(J)} c(\alpha) Z^{\alpha}.$$

We set

$$S_{n,J}(c, Z) = \sum_{m=1}^n \Phi_{m,J}(c, Z) = \sum_{m=1}^n m! \Phi_{m,J}^o(c, Z)$$

and we prove that for every $n, J \in \mathbb{N}$

$$\|S_{n,J}(c, Z)\|_{\mathcal{U},p}^2 \leq \sum_{m=1}^n M_p^{2m} m! |c|_{\mathcal{U},m,J}^2 \leq N_0^2(c, M_p). \quad (5.8)$$

Then (5.7) follows by passing to the limit. We prove (5.8) by recurrence on n . For $n = 1$, we use the Burkholder inequality (A.2) and we have

$$\|S_{1,J}(c, Z)\|_{\mathcal{U},p}^2 = \left\| \sum_{j=1}^J c_j Z_j \right\|_{\mathcal{U},p}^2 \leq b_p^2 \sum_{j=1}^J \|c_j Z_j\|_{\mathcal{U},p}^2 \leq b_p^2 M_p(Z)^2 \sum_{j=1}^J |c_j|_{\mathcal{U}}^2 \leq M_p^2 |c|_{\mathcal{U},1,J}^2.$$

For $n > 1$, we use the following basic decomposition:

$$\Phi_{m,J}^o(c, Z) = \sum_{j=1}^J Z_j \sum_{\alpha \in \Gamma_{m-1}^o(j-1)} c(\alpha, j) Z^\alpha.$$

This gives

$$\begin{aligned} S_{n,J}(c, Z) &= \sum_{m=1}^n m! \Phi_{m,J}^o(c, Z) = \sum_{j=1}^J Z_j \sum_{m=1}^n m! \sum_{\alpha \in \Gamma_{m-1}^o(j-1)} c(\alpha, j) Z^\alpha \\ &= \sum_{j=1}^J Z_j c_j + \sum_{j=1}^J Z_j \sum_{m=2}^n \sum_{\alpha \in \Gamma_{m-1}(J)} (m 1_{\Gamma(j-1)}(\alpha) c(\alpha, j)) Z^\alpha \\ &= \sum_{j=1}^J Z_j (c_j + S_{n-1,J}(c^j)) \end{aligned}$$

with

$$c^j(\alpha) = (1 + |\alpha|) \times 1_{\Gamma(j-1)}(\alpha) c(\alpha, j) \quad \alpha \in \Gamma_{m-1}(J).$$

Notice that $S_{n-1,J}(c^j)$ is measurable with respect to $\sigma(Z_1, \dots, Z_{j-1})$ so that $S_{n,J}(c, Z)$ is a martingale. Then, by Burkholder's inequality (A.2),

$$\begin{aligned} \|S_{n,J}(c, Z)\|_{\mathcal{U},p}^2 &\leq b_p^2 \sum_{j=1}^J \|Z_j\|_p^2 \|c_j + S_{n-1,J}(c^j, Z)\|_{\mathcal{U},p}^2 \\ &\leq 2b_p^2 M_p^2(Z) \left(\sum_{j=1}^J |c_j|_{\mathcal{U}}^2 + \sum_{j=1}^J \|S_{n-1,J}(c^j, Z)\|_{\mathcal{U},p}^2 \right). \end{aligned}$$

Using the recurrence hypothesis

$$\begin{aligned} \sum_{j=1}^J \|S_{n-1,J}(c^j, Z)\|_{\mathcal{U},p}^2 &\leq \sum_{j=1}^J \sum_{m=1}^{n-1} M_p^{2m} m! |c^j|_{\mathcal{U},m,J}^2 \\ &= \sum_{m=1}^{n-1} M_p^{2m} m! (m+1)^2 \sum_{j=1}^J \sum_{\alpha \in \Gamma_m(j-1)} |c(\alpha, j)|_{\mathcal{U}}^2 \end{aligned}$$

Since

$$|c|_{\mathcal{U},m+1,J}^2 = (m+1) \sum_{j=1}^J \sum_{\alpha \in \Gamma_m(j-1)} |c(\alpha, j)|_{\mathcal{U}}^2$$

we obtain

$$\sum_{j=1}^J \|S_{n-1,J}(c^j, Z)\|_{\mathcal{U},p}^2 \leq \sum_{m=1}^{n-1} M_p^{2m} (m+1)! |c|_{\mathcal{U},m+1,J}^2.$$

We conclude that

$$\|S_{n,J}(c, Z)\|_{\mathcal{U},p}^2 \leq M_p^2 \left(|c|_{\mathcal{U},1}^2 + \sum_{m=2}^n M_p^{2(m-1)} m! |c|_{\mathcal{U},m,J}^2 \right) = \sum_{m=1}^n M_p^{2m} m! |c|_{\mathcal{U},m,J}^2.$$

□

We estimate now the derivatives of $S(c, Z)$.

Proposition 5.3 *We assume that (5.4) holds and, for $p \geq 2$, set M_p as in (5.6). For $q \in \mathbb{N}$ one has*

$$\|D^{(q)}S(c, Z)\|_{\mathcal{H}(\mathcal{U})^{\otimes q},p} \leq \sqrt{2} \left(\sum_{n=q}^{\infty} n! \frac{n!}{(n-q)!} M_p^{2(n-q)} |c|_{\mathcal{U},n}^2 \right)^{1/2} = \sqrt{2} N_q(c, M_p^2).$$

As a consequence,

$$\|S(c, Z)\|_{\mathcal{U},q,p} \leq \sqrt{2} \sum_{k=0}^q N_k(c, M_p^2). \quad (5.9)$$

Proof. Let us denote $\partial_j \Phi_m(c, Z) = \partial_{Z_j} \Phi_m(c, Z)$ so that $D_j \Phi_m(c, Z) = \chi_j \partial_j \Phi_m(c, Z)$. We write

$$\partial_j \Phi_m(c, Z) = \sum_{\beta \in \Gamma_m} c(\beta) \partial_j Z^\beta = m \sum_{\substack{\alpha \in \Gamma_{m-1} \\ j \notin \alpha}} c(\alpha, j) Z^\alpha = \Phi_{m-1}(\bar{c}^j, Z)$$

with

$$\bar{c}^j(\alpha) = (1 + |\alpha|) c(\alpha, j) 1_{\{j \notin \alpha\}}.$$

So we may write

$$\partial_j S(c, Z) = \sum_{m=1}^{\infty} \partial_j \Phi_{m,J}(c, Z) = c_j + \sum_{m=2}^{\infty} \Phi_{m-1,J}(\bar{c}^j, Z).$$

We set $\bar{c}_0 = (c_j)_{j \in \mathbb{N}}$ and $\bar{c}_\alpha = (\bar{c}_\alpha^j)_{j \in \mathbb{N}}$. Notice that $\bar{c}_0, \bar{c}_\alpha \in \mathcal{H}(\mathcal{U})$, so the above equality reads

$$\nabla S(c, Z) = \bar{c}_0 + \sum_{m=1}^{\infty} \Phi_{m,J}(\bar{c}, Z) = \bar{c}_0 + S(\bar{c}, Z) \in \mathcal{H}(\mathcal{U}) \quad (5.10)$$

where $\nabla S(c, Z) = (\partial_j S(c, Z))_{j \in \mathbb{N}} \in \mathcal{H}(\mathcal{U})$. By using (5.7),

$$\|\nabla S(c, Z)\|_{\mathcal{H}(\mathcal{U}),p}^2 \leq 2(|\bar{c}_0|_{\mathcal{H}(\mathcal{U})}^2 + \|S(\bar{c}, Z)\|_{\mathcal{H}(\mathcal{U}),p}^2) \leq 2 \left(|\bar{c}(0)|_{\mathcal{H}(\mathcal{U})}^2 + \sum_{m=1}^{\infty} M_p^{2m} m! |\bar{c}|_{\mathcal{H}(\mathcal{U}),m}^2 \right) \quad (5.11)$$

We have $|\bar{c}_0|_{\mathcal{H}(\mathcal{U})}^2 = \sum_{j=1}^{\infty} |c_j|_{\mathcal{U}}^2 = |c|_{\mathcal{U},1}^2$ and

$$\begin{aligned} |\bar{c}|_{\mathcal{H}(\mathcal{U}),m}^2 &= \sum_{j=1}^{\infty} \sum_{\alpha \in \Gamma_m} |\bar{c}^j(\alpha)|_{\mathcal{U}}^2 = \sum_{j=1}^{\infty} \sum_{\alpha \in \Gamma_m} (1 + |\alpha|)^2 |c(\alpha, j)|_{\mathcal{U}}^2 1_{\{j \notin \alpha\}} \\ &= (m+1)^2 \sum_{\beta \in \Gamma_{m+1}} |c(\beta)|_{\mathcal{U}}^2 = (m+1)^2 |c|_{\mathcal{U},m+1}^2. \end{aligned} \quad (5.12)$$

By inserting in (5.11), this gives

$$\begin{aligned} \|\nabla S(c, Z)\|_{\mathcal{H}(\mathcal{U}),p}^2 &\leq 2 \left(|c|_{\mathcal{U},1}^2 + \sum_{m=1}^{\infty} M_p^{2m} m! (m+1)^2 |c|_{\mathcal{U},m+1}^2 \right) \\ &= 2 \left(|c|_{\mathcal{U},1}^2 + \sum_{m=1}^{\infty} M_p^{2m} (m+1)! (m+1) |c|_{\mathcal{U},m+1}^2 \right) \\ &\leq 2 \sum_{m=1}^{\infty} M_p^{2m} m! m |c|_{\mathcal{U},m}^2 = 2N_1(c, M_p^2)^2, \end{aligned}$$

that is $\|\nabla S(c, Z)\|_{\mathcal{H}(\mathcal{U}),p} \leq \sqrt{2} N_1(c, M_p^2)$. Finally, one has $D_k S(c, Z) = \chi_k \partial_j S(c, Z)$ and since $\chi_k \in [0, 1]$ then $|DS(c, Z)|_{\mathcal{H}(\mathcal{U})} \leq |\nabla S(c, Z)|_{\mathcal{H}(\mathcal{U})}$. So, $\|DS(c, Z)\|_{\mathcal{H}(\mathcal{U}),p} \leq \|\nabla S(c, Z)\|_{\mathcal{H}(\mathcal{U}),p}$ and the proof is completed for the first order derivative.

Let us now estimate the second order derivatives. We have $DD\Phi_{m,J}(c, Z) = D\Phi_{m,J}(\bar{c}, Z)$. Using (5.12) one checks that

$$|\bar{c}|_{\mathcal{H}(\mathcal{U}),1} + N_1(\bar{c}, M_p^2) \leq \sqrt{2} N_2(c, M_p^2)$$

and so we may use the result for the first order derivatives. For higher order derivatives the argument is the same. \square

We estimate now the Sobolev norms of $LS(c, Z)$.

Proposition 5.4 *We assume that (5.4) holds and, for $p \geq 2$, set M_p as in (5.6). For every $q \in \mathbb{N}$ there exists a constant C (depending on q and p only) such that*

$$\|LS(c, Z)\|_{\mathcal{U},q,p} \leq \frac{C}{r^{q+1}} \left(\sum_{n=1}^{q+1} |c|_{\mathcal{U},n} + N_{q+2}(c, M_p^2) \right). \quad (5.13)$$

Proof. Notice that $\langle DZ_k, DZ_j \rangle = 0$ for $k \neq j$. Using the computational rules one obtains for $\beta = (\beta_1, \dots, \beta_m)$ with $m \geq 2$

$$LZ^\beta = \sum_{k=1}^m LZ_{\beta_k} \prod_{j \neq k} Z_{\beta_j}$$

and, using the symmetry of $c(\beta)$, this gives

$$\begin{aligned} L\Phi_m(c, Z) &= m! L\Phi_m^o(c, Z) = m! m \sum_{j=1}^{\infty} LZ_j \sum_{\alpha \in \Gamma_{m-1}^o(j-1)} c(\alpha, j) Z^\alpha \\ &= m^2 \sum_{j=1}^{\infty} LZ_j \sum_{\alpha \in \Gamma_{m-1}(j-1)} c(\alpha, j) Z^\alpha = \sum_{j=1}^{\infty} LZ_j \Phi_{m-1}(\tilde{c}^j, Z) \end{aligned}$$

with

$$\tilde{c}^j(\alpha) = (1 + |\alpha|)^2 \times 1_{\max_k \alpha_k \leq j-1} c(\alpha, j).$$

For $m = 1$ we have $L\Phi_1(c, Z) = \sum_{j=1}^{\infty} c_j LZ_j$. It follows that, for $S_n(c, Z) = \sum_{m=1}^n \Phi_m(c, Z)$,

$$\begin{aligned} LS_n(c, Z) &= \sum_{m=1}^n L\Phi_m(c, Z) = \sum_{j=1}^{\infty} c_j LZ_j + \sum_{j=1}^{\infty} LZ_j \sum_{m=2}^n \Phi_{m-1}(\tilde{c}^j, Z) \\ &= \sum_{j=1}^{\infty} LZ_j (c_j + S_{n-1}(\tilde{c}^j, Z)). \end{aligned}$$

Notice that $S_{n-1}(\tilde{c}^j, Z)$ is $\sigma(Z_1, \dots, Z_{j-1})$ measurable. Since $LS_n(c)$ verifies (A.3) with $B_j = c_j + S_{n-1}(\tilde{c}^j, Z)$ and $\Lambda_m = 0$, we will use (A.4) (actually, we should use $S_{n,J}(c, Z)$ and then pass to the limit following the standard technique). We have

$$\begin{aligned} \sum_{j=1}^{\infty} \|B_j\|_{\mathcal{U},q,p}^2 &\leq 2 \left(|c|_{\mathcal{U},1}^2 + \sum_{j=1}^{\infty} \|S_{n-1}(\tilde{c}^j, Z)\|_{\mathcal{U},q,p}^2 \right) \\ &\leq C \left(|c|_{\mathcal{U},1}^2 + \sum_{j=1}^{\infty} \sum_{\ell=1}^q \left\| D^{(\ell)} S_{n-1}(\tilde{c}^j, Z) \right\|_{\mathcal{H}(\mathcal{U})^{\otimes \ell},p}^2 \right). \end{aligned}$$

By using (5.9) we obtain

$$\sum_{j=1}^{\infty} \|B_j\|_{U,q,p}^2 \leq C \left(|c|_{\mathcal{U},1}^2 + \sum_{j=1}^{\infty} \sum_{\ell=1}^q \sum_{n=\ell}^{\infty} n! \frac{n!}{(n-\ell)!} M_p^{2(n-\ell)} |\tilde{c}^j|_{\mathcal{U},n}^2 \right) =: C(|c|_{\mathcal{U},1}^2 + I).$$

In order to handle I , we compute

$$\begin{aligned} \sum_{j=1}^{\infty} |\tilde{c}^j|_{\mathcal{U},n}^2 &= (n+1)^4 \sum_{j=1}^{\infty} \sum_{\alpha \in \Gamma_n(j-1)} |c(\alpha, j)|_{\mathcal{U}}^2 = (n+1)^4 n! \sum_{j=1}^{\infty} \sum_{\alpha \in \Gamma_n^o(j-1)} |c(\alpha, j)|_{\mathcal{U}}^2 \\ &= (n+1)^4 n! \sum_{\beta \in \Gamma_{n+1}^o} |c(\beta)|_{\mathcal{U}}^2 = (n+1)^3 \sum_{\beta \in \Gamma_{n+1}} |c(\beta)|_{\mathcal{U}}^2 = (n+1)^3 |c|_{\mathcal{U},n+1}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} I &= \sum_{\ell=1}^q \sum_{n=\ell}^{\infty} M_p^{2(n-\ell)} (n+1)! \frac{(n+1)!}{(n-\ell)!} (n+1) |c|_{\mathcal{U},n+1}^2 \\ &= \sum_{\ell=1}^q \left(((\ell+1)!)^2 (\ell+1) |c|_{\mathcal{U},\ell+1}^2 + \sum_{m=\ell+2}^{\infty} M_p^{2(m-1-\ell)} m! \frac{m!}{(m-1-\ell)!} m |c|_{\mathcal{U},m}^2 \right) \\ &\leq C \sum_{\ell=1}^q (|c|_{\mathcal{U},\ell+1}^2 + N_{\ell+1}^2(c, M_p^2)) \end{aligned}$$

Now, for $k \leq q+2$ straightforward computations give

$$N_k(c, M)^2 \leq k! |c|_{\mathcal{U},k}^2 + M N_{k+1}(c, M)^2 \leq \dots \leq C \left(\sum_{i=k}^{q+1} |c|_{\mathcal{U},i}^2 + N_{q+2}(c, M)^2 \right)$$

and by inserting we obtain

$$I \leq C \left(\sum_{n=2}^{q+1} |c|_{\mathcal{U},n}^2 + N_{q+2}(c, M_p^2) \right).$$

By resuming,

$$\sum_{j=1}^{\infty} \|B_j\|_{U,q,p}^2 \leq C \left(\sum_{n=1}^{q+1} |c|_{\mathcal{U},n}^2 + N_{q+2}(c, M_p^2) \right)$$

and by applying (A.4) we get

$$\|LS(c, Z)\|_{\mathcal{U},q,p} \leq \frac{C}{r^{q+1}} \left(\sum_{j=1}^{\infty} \|B_j\|_{U,q,p}^2 \right)^{1/2} \leq \frac{C}{r^{q+1}} \left(\sum_{n=1}^{q+1} |c|_{\mathcal{U},n}^2 + N_{q+2}(c, M_p^2) \right)$$

and the statement is proved. \square

As an immediate consequence of Proposition 5.3 and 5.4 we obtain:

Proposition 5.5 *We assume that (5.4) holds and, for $p \geq 2$, set M_p as in (5.6). For every $q \geq 2$ there exists a constant $C \geq 1$ depending on p, q such that*

$$\| |S(c, Z)| \|_{q,p} \equiv \|S(c, Z)\|_{q,p} + \|LS(c, Z)\|_{q-2,p} \leq \frac{C}{r^{q-1}} \left(\sum_{n=1}^{q-1} |c|_{\mathcal{U},n}^2 + N_q(c, M_p^2) \right). \quad (5.14)$$

We conclude this section with a result concerning the non degeneracy of the Malliavin covariance matrix of $S(c, Z)$. Actually we are not able to obtain such estimates for general series but for finite series only.

Lemma 5.6 *For $N \in \mathbb{N}$, set*

$$S_N(c, Z) = \sum_{1 \leq |\alpha| \leq N} c(\alpha) Z^\alpha \quad \text{and} \quad i_N(c) = \sum_{m=1}^N \sum_{\alpha \in \Gamma_m} m! c(\alpha)^2.$$

Then for every $p \geq 1$ such that $\sup_k \|Z_k\|_{2p} < \infty$ there exists a universal constant C_p such that for every $\eta \leq \frac{1}{2} m(r) i_N(c)$ we have

$$\mathbb{P}(\sigma_{S_N(c, Z)} \leq \eta) \leq \left(\frac{C_p(1 + i_N(c))}{m(r) i_N(c)} (N!)^3 2^N N^{-5/4} (\bar{\kappa}_N(c) + \bar{\delta}_N(c)) \right)^p \quad (5.15)$$

with

$$\bar{\kappa}_N(c) = \sum_{l=1}^N \kappa_{4,l}^{1/4}(c) \quad \text{and} \quad \bar{\delta}_N(c) = \sum_{l=1}^N \delta_l(c). \quad (5.16)$$

Proof. We write

$$\begin{aligned} \sigma_{S_N(c, Z)} &= \sum_{j=1}^{\infty} |\partial_j S_N(c, Z)|^2 \chi_j = \sum_{j=1}^{\infty} |\partial_j S_N(c, Z)|^2 \tilde{\chi}_j + m(r) \sum_{j=1}^{\infty} |\partial_j S_N(c, Z)|^2 \\ &= \sum_{j=1}^{\infty} |\partial_j S_N(c, Z)|^2 \tilde{\chi}_j + m(r) \left(\sum_{j=1}^{\infty} |\partial_j S_N(c, Z)|^2 - i_N(c) \right) + m(r) i_N(c). \end{aligned}$$

We set

$$\tilde{I}_N(c, Z) = \sum_{j=1}^{\infty} |\partial_j S_N(c, Z)|^2 \tilde{\chi}_j \quad \text{and} \quad I_N(c, Z) = \sum_{j=1}^{\infty} |\partial_j S_N(c, Z)|^2.$$

For $\eta \leq \frac{m(r) i_N(c)}{2}$, it follows that

$$\begin{aligned} \mathbb{P}(\sigma_{S_N(c, Z)} \leq \eta) &= \mathbb{P}\left(\tilde{I}_N(c, Z) + m(r)(I_N(c, Z) - i_N(c)) \leq -\frac{m(r) i_N(c)}{2}\right) \\ &\leq \mathbb{P}\left(\tilde{I}_N(c, Z) \leq -\frac{m(r) i_N(c)}{4}\right) + \mathbb{P}\left(m(r)(I_N(c, Z) - i_N(c)) \leq -\frac{m(r) i_N(c)}{4}\right) \\ &\leq \mathbb{P}\left(|\tilde{I}_N(c, Z)| \geq \frac{m(r) i_N(c)}{4}\right) + \mathbb{P}\left(|I_N(c, Z) - i_N(c)| \geq \frac{i_N(c)}{4}\right) \\ &\leq \left(\frac{4}{m(r) i_N(c)}\right)^p (\|\tilde{I}_N(c, Z)\|_p^p + \|I_N(c, Z) - i_N(c)\|_p^p) \end{aligned}$$

Now, the real difficulty is to produce L^p estimates for $\tilde{I}_N(c, Z)$ and $I_N(c, Z) - i_N(c)$. Section B.2 in Appendix B is devoted to such a problem, and the final result is given in Lemma B.8. So, we use (B.30) and the statement immediately follows. \square

6 Convergence in total variation

In this section we study the convergence of stochastic series to the Gaussian law in two situations: first, we consider finite series and we obtain estimates of the error which are not asymptotic; in a second stage we deal with infinite series and we prove a convergence result, but in this case we are no more able to get the rate of convergence.

6.1 Error estimates for finite series

The aim of this section is to obtain non asymptotic estimates for the invariance principle in total variation distance. We stress that here N is finite and fixed and that we consider a fixed set of coefficients $c = (c(\alpha))_\alpha$ - so the results is not asymptotic. The estimates will be given in terms of $\bar{\kappa}_N(c)$ defined in (2.12) and of $\bar{\delta}_N(c) = \sum_{l=1}^N l! \delta_l(c)$ with $\delta_N(c)$ defined in (2.5). We will use the normalization hypothesis

$$i_N(c) = \sum_{m=1}^N \sum_{\alpha \in \Gamma_m} m! c(\alpha)^2 = 1 \quad (6.1)$$

and

$$\sum_{\ell=1}^N \ell! \delta_\ell(c)^2 \leq \frac{i_N(c)}{4} = \frac{1}{4} \quad (6.2)$$

Given two sequences $(Z_k)_{k \in \mathbb{N}}$ and $(\bar{Z}_k)_{k \in \mathbb{N}}$ we denote

$$M_p = M_p(Z, \bar{Z}) = \max_k \|Z_k\|_p \vee \|\bar{Z}_k\|_p.$$

The main result in this section is the following:

Theorem 6.1 *Let $(Z_k)_{k \in \mathbb{N}}$ and $(\bar{Z}_k)_{k \in \mathbb{N}}$ satisfy (5.4) and such that $Z_k \in \mathcal{L}(r, \varepsilon)$, $\bar{Z}_k \in \mathcal{L}(r, \varepsilon)$ and $M_p = M_p(Z, \bar{Z}) < \infty$ for every $p \geq 1$. We also assume that (6.1) and (6.2) hold true. Then for every $p_* \geq 1$ there exist positive constant C_* , d_* , c_* and M_* such that for every $N \in \mathbb{N}$*

$$|\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(S_N(c, \bar{Z})))| \leq \frac{C_*}{(m(r)r)^{d_*}} M_*^N N^{c_*} (N!)^{3p_*} \|f\|_\infty (\bar{\kappa}_N(c)^{p_*} + \bar{\delta}_N(c)). \quad (6.3)$$

We stress that all constants depend on the random sequences $(Z_k)_{k \in \mathbb{N}}$ and $(\bar{Z}_k)_{k \in \mathbb{N}}$ only through $M_p = M_p(Z, \bar{Z})$ for a suitably large p .

Proof. We first give some estimates which are specific to finite series (under the hypothesis that $c(\alpha) = 0$ if $|\alpha| \geq N$). First of all, for $q \geq 1$,

$$\begin{aligned} N_q(c, M) &= \left(\sum_{m=q}^N M^{m-q} \times \frac{(m!)^2}{(m-q)!} \sum_{\alpha \in \Gamma_m} c(\alpha)^2 \right)^{1/2} \leq M^{(N-q)/2} N^{q/2} \left(\sum_{m=q}^N m! \sum_{\alpha \in \Gamma_m} c(\alpha)^2 \right)^{1/2} \\ &\leq M^{(N-q)/2} N^{q/2} \end{aligned} \quad (6.4)$$

the last inequality being true if $i_N(c) \leq 1$. As an immediate consequence of this and of (5.14), for every $p \geq 1$

$$\|S_N(c, Z)\|_{q,p} \leq \frac{C_p}{r^{q-1}} \times M_p^{2N} N^{q/2}. \quad (6.5)$$

Moreover let $\tilde{c}^k(\alpha) = c(k, \alpha)$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} N_q(\tilde{c}^k, M) &\leq M^{3(N-q)/2} N^{3q/2} \sum_{k=1}^{\infty} \left(\sum_{m=q}^N m! \sum_{\alpha \in \Gamma_m} c(k, \alpha)^2 \right)^{3/2} \\ &\leq M^{3(N-q)/2} N^{3q/2} \bar{\delta}_N(c) \sum_{k=1}^{\infty} \left(\sum_{m=q}^N m! \sum_{\alpha \in \Gamma_m} c(k, \alpha)^2 \right) \\ &\leq M^{3(N-q)/2} N^{3q/2} \bar{\delta}_N(c). \end{aligned} \quad (6.6)$$

We are now able to start the proof itself.

Step 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $\|f\|_\infty \leq 1$. For $\delta > 0$, let f_δ denote its regularization, as in (4.33). Then we have

$$|\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(S_N(c, \bar{Z})))| \leq a_N(\delta) + b_N(\delta) + \bar{b}_N(\delta) \quad (6.7)$$

in which

$$\begin{aligned} a_N(\delta) &= |\mathbb{E}(f_\delta(S_N(c, Z))) - \mathbb{E}(f_\delta(S_N(c, \bar{Z})))|, \\ b_N(\delta) &= |\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f_\delta(S_N(c, Z)))|, \quad \bar{b}_N(\delta) = |\mathbb{E}(f(S_N(c, \bar{Z}))) - \mathbb{E}(f_\delta(S_N(c, \bar{Z})))|. \end{aligned}$$

So, we study separately such contributions.

Step 2: estimate of $a_N(\delta)$. We use some facts developed in the proof of Theorem 3.1. Let $J \in \mathbb{N}$ and

$$S_{N,J}(c, Z) = \sum_{n=1}^N \sum_{\alpha \in \Gamma_n(J)} c(\alpha) Z^\alpha.$$

We also set

$$a_{N,J}(\delta) = |\mathbb{E}(f_\delta(S_{N,J}(c, Z))) - \mathbb{E}(f_\delta(S_{N,J}(c, \bar{Z})))|.$$

Since the estimate for $a_{N,J}$ will not depend on J , we will get the result for $a_N(\delta)$ by passing to the limit. So, we recall the following facts.

For $\theta \in (0, 1)$, in Theorem 3.1 we have denoted

$$\hat{Z}^k(\theta) = (Z_1, \dots, Z_{k-1}, \theta Z_k, \bar{Z}_{k+1}, \dots, \bar{Z}_J), \quad k = 0, 1, \dots, J,$$

with $\hat{Z}^0(\theta) = (\bar{Z}_1, \dots, \bar{Z}_J)$ and $\hat{Z}^J(\theta) = (Z_1, \dots, Z_{J-1}, \theta Z_J)$. Moreover we have denoted

$$s_k = S_{N,J}(c, \hat{Z}^k(0)) \quad \text{and} \quad v_k = c(k) + \sum_{n=2}^N \sum_{\substack{\beta \in \Gamma_{n-1}(J) \\ k \notin \beta}} c(k, \beta) (\hat{Z}^k(0))^\beta$$

so that

$$S_{N,J}(c, \hat{Z}^k(\theta)) = s_k + \theta Z_k v_k$$

and we have proved that

$$\begin{aligned} \mathbb{E}(f_\delta(S_{N,J}(c, Z))) - \mathbb{E}(f_\delta(S_{N,J}(c, \bar{Z}))) &= \frac{1}{6} \sum_{k=1}^J \int_0^1 \mathbb{E}(f_\delta'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3) \theta d\theta \\ &\quad - \frac{1}{6} \sum_{k=1}^J \int_0^1 \mathbb{E}(f_\delta'''(s_k + \theta \bar{Z}_k v_k) \bar{Z}_k^3 v_k^3) \theta d\theta. \end{aligned}$$

In the original proof of Theorem 3.1 we upper bounded $f_\delta'''(s_k + \theta Z_k v_k)$ by $\|f_\delta'''\|_\infty$ but now we will use integration by parts in order to get rid of the derivatives. In order to do it we will use Lemma 4.5 with $F_k = s_k + \theta Z_k v_k$ and $G_k = Z_k^3 v_k^3$, $M = 1$ and $q = 3$. Then we apply (4.35) with $p_1 = p_2 = p_3 = 3$: for every $\eta_k > 0$ we obtain

$$|\mathbb{E}(f_\delta'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3)| \leq C \|f\|_\infty \lambda_{\delta, \eta_k, 9}^3(s_k + \theta Z_k v_k) (1 + \|s_k + \theta Z_k v_k\|_{1,5,36}^{12}) \|Z_k^3 v_k^3\|_{3,9} \quad (6.8)$$

with (see (4.31))

$$\lambda_{\delta, \eta_k, 9}(s_k + \theta Z_k v_k) = \frac{1}{\delta} \mathbb{P}(\lambda_{s_k + \theta Z_k v_k} \leq \eta_k)^{\frac{1}{9}} + \frac{1}{\eta_k}.$$

We denote

$$c^{k,\theta}(\alpha) = c(\alpha)(1_{\{k \notin \alpha\}} + \theta 1_{\{k \in \alpha\}}).$$

Then

$$s_k + \theta Z_k v_k = S_{N,J}(c, \widehat{Z}^k(\theta)) = S_{N,J}(c^{k,\theta}, \widehat{Z}^k(1)).$$

So, we take $\eta_k = m(r)i_{m,J}(c^{k,\theta})$, with $i_{N,J}(c^{k,\theta}) = \sum_{\ell=1}^N \sum_{\alpha \in \Gamma_\ell(J)} \ell! c^{k,\theta}(\alpha)^2$, and we use (5.15) with $\bar{p} \geq 1$: we get

$$\mathbb{P}(\lambda_{s_k + \theta Z_k v_k} \leq \eta_k) \leq \left(\frac{C_{p,N}}{m(r)i_{N,J}(c^{k,\theta})} \right)^{\bar{p}} \bar{\kappa}_N(c^{k,\theta})^{\bar{p}}$$

for every $\theta \in (0, 1)$, where $C_{p,N} = C_p \mathcal{D}_N$ is given in Lemma 5.6:

$$\mathcal{D}_N = (N!)^3 2^N N^{-5/4}.$$

Now, observe that

$$\begin{aligned} i_{N,J}(c^{k,\theta}) &= \sum_{\ell=1}^N \sum_{\alpha \in \Gamma_\ell(J)} \ell! c^{k,\theta}(\alpha)^2 = \sum_{\ell=1}^N \sum_{\alpha \in \Gamma_\ell(J)} \ell! c(\alpha)^2 - (1-\theta) \sum_{\ell=1}^N \sum_{\alpha \in \Gamma_\ell(J)} \ell! c(\alpha)^2 \mathbf{1}_{k \in \alpha} \\ &\geq i_{N,J}(c) - (1-\theta) \sum_{\ell=1}^N \ell! \delta_\ell(c)^2. \end{aligned}$$

Under (6.2), for every $\theta \in (0, 1)$ and for every J large, one can write

$$i_{N,J}(c^{k,\theta}) \geq \frac{i_N(c)}{2} - \frac{i_N(c)}{4} = \frac{1}{4}.$$

One also has

$$\bar{\kappa}_N(c^{k,\theta}) \leq 2\bar{\kappa}_N(c) \quad \text{and} \quad \bar{\delta}_N(c^{k,\theta}) \leq 2\bar{\delta}_N(c),$$

so finally

$$\mathbb{P}(\lambda_{s_k + \theta Z_k v_k} \leq \eta_k) \leq \left(\frac{C_{\bar{p},N}}{m(r)} \right)^{\bar{p}} (\bar{\kappa}_N(c) + \bar{\delta}_N(c))^{\bar{p}}.$$

Moreover, for every $k \in \mathbb{N}$, $\theta \in (0, 1)$ and for every J large we have

$$\frac{1}{\eta_k} \leq \frac{16}{m(r)}$$

and then

$$\lambda_{\delta, \eta_k, 9}(s_k + \theta Z_k v_k) = \frac{1}{\delta} \left(\frac{C_{\bar{p},N}}{m(r)} \right)^{\frac{\bar{p}}{9}} \bar{\kappa}_N(c)^{\frac{\bar{p}}{9}} + \frac{16}{m(r)}.$$

Now we choose

$$\delta = \delta_{\bar{p}} = \left(\frac{C_{\bar{p},N}}{m(r)} \right)^{\frac{\bar{p}}{9}} (\bar{\kappa}_N(c) + \bar{\delta}_N(c))^{\frac{\bar{p}}{9}} \quad (6.9)$$

and then

$$\lambda_{\delta_{\bar{p}}, \eta_k, 9}(s_k + \theta Z_k v_k) \leq 1 + \frac{16}{m(r)}.$$

Coming back to (6.8), we have

$$\left| \mathbb{E}(f_{\delta_{\bar{p}}}'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3) \right| \leq \frac{C}{m^3(r)} \|f\|_\infty (1 + \|s_k + \theta Z_k v_k\|_{1,5,36}^{12}) \|Z_k^3 v_k^3\|_{3,9}. \quad (6.10)$$

We recall that $s_k + \theta Z_k v_k = S_{N,J}(c^{k,\theta}, \widehat{Z}^k(1))$ and we use (6.5) with c replaced by $c^{k,\theta}$ and we obtain

$$\|s_k + \theta Z_k v_k\|_{1,5,36} \leq \frac{C}{r^4} M_{36}^N N^{5/2}.$$

Since Z_k and v_k are independent we have

$$\|Z_k^3 v_k^3\|_{3,9} = \|Z_k^3\|_{3,9} \|v_k^3\|_{3,9} \leq CM_{27}^3 \|v_k\|_{3,27}^3$$

the last inequality being true because of (4.30). Summing on k in (6.10) we get

$$\sum_{k=1}^J \left| \mathbb{E}(f_{\delta_{\overline{p}}}'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3) \right| \leq \frac{CM_{36}^{12N} N^{30}}{m^3(r) r^{48}} \|f\|_{\infty} \sum_{k=1}^J \|v_k\|_{3,27}^3.$$

Since $v_k = c(k) + S_{m-1,J}(\tilde{c}^k, \widehat{Z}^k(0))$ with $\tilde{c}^k(\alpha) = c(k, \alpha)$, we use (5.9) and (6.6) and we obtain

$$\begin{aligned} \sum_{k=1}^J \|v_k\|_{3,27}^3 &\leq C \sum_{k=1}^J \left(\sum_{l=0}^3 N_l(\tilde{c}^k, M_{27}^2) \right)^3 \\ &\leq C \sum_{l=0}^3 \sum_{k=1}^J N_l^3(\tilde{c}^k, M_{27}^2) \\ &\leq CM_{27}^{3N} N^{9\overline{\delta}_N(c)}. \end{aligned}$$

Inserting in the previous inequality we get

$$\sum_{k=1}^J \left| \mathbb{E}(f_{\delta_{\overline{p}}}'''(s_k + \theta Z_k v_k) Z_k^3 v_k^3) \right| \leq \frac{CM_{27}^{15N} N^{39}}{m^3(r) r^{48}} \|f\|_{\infty} \overline{\delta}_N(c).$$

Since the same estimate holds with Z replaced by \overline{Z} we conclude that

$$a_N(\delta_{\overline{p}}) \leq \frac{CM_{27}^{15N} N^{39}}{m^3(r) r^{48}} \|f\|_{\infty} \overline{\delta}_N(c). \quad (6.11)$$

Step 3: estimate of $b(\delta_{\overline{p}})$ and $\overline{b}(\delta_{\overline{p}})$. We use the regularization inequality (4.36) in Lemma 4.6 with $\eta = \frac{1}{2}m(r)i_N(c) = \frac{1}{2}m(r)$ and we obtain:

$$b_N(\delta) \leq C_* \|f\|_{\infty} \left(\mathbb{P}\left(\lambda_{S_N(c,Z)} \leq \frac{m(r)}{2}\right) + \frac{\sqrt{\delta}}{m(r)^{l_*}} (1 + \| |S(c, Z)| \|_{3,l_*})^{a_*} \right)$$

where C_*, l_*, a_* are universal constants. We take now $\delta = \delta_{\overline{p}}$ as in (6.9) and we use (5.15), so that

$$b_N(\delta_{\overline{p}}) \leq C_* \|f\|_{\infty} \left(\left(\frac{C_{\widehat{p}} \mathcal{D}_N}{m(r)} (\overline{\kappa}_N(c) + \overline{\delta}_N(c)) \right)^{\widehat{p}} + \frac{(\frac{C_{\overline{p}} \mathcal{D}_N}{m(r)} (\overline{\kappa}_N(c) + \overline{\delta}_N(c)))^{\frac{\overline{p}}{18}}}{m(r)^{l_*}} (1 + \| |S_N(c, Z)| \|_{3,l_*})^{a_*} \right).$$

By applying (6.5) and taking $\widehat{p} = p_*$ and $\overline{p} = 18p_*$, with $p_* \geq 1$, we get

$$\begin{aligned} b_N(\delta_{\overline{p}}) &\leq C_* \|f\|_{\infty} \left(\frac{\mathcal{D}_N}{m(r)} (\overline{\kappa}_N(c) + \overline{\delta}_N(c)) \right)^{p_*} \times \left(\frac{1}{r^2} \times M_{l_*}^N N^{3/2} \right)^{a_*} \\ &\leq \frac{C_*}{(m(r)r)^{d_*}} M_*^N N^{c_*} \mathcal{D}_N^{p_*} \|f\|_{\infty} (\overline{\kappa}_N(c) + \overline{\delta}_N(c))^{p_*}. \end{aligned}$$

The same estimate holds for $\overline{b}(\delta_{\overline{p}})$. This together with (6.11) and (6.7) yield (6.3). \square

6.2 Gaussian limit

In this section we estimate the total variation distance between $S_N(c, Z)$ and a standard normal distributed random variable G . This will be an immediate consequence of the result from the previous section and the following theorem due to Nourdin and Peccati [17].

Theorem 6.2 *Let $(\bar{Z}_k)_{k \in \mathbb{N}}$ with \bar{Z}_k standard normal random variables and let G be another standard normal random variable. Suppose that (6.1) holds. There exists an universal constant C such that for every N and every measurable and bounded function f*

$$|\mathbb{E}(f(S_N(c, \bar{Z}))) - \mathbb{E}(f(G))| \leq 3 \|f\|_\infty N^3 (2N)! N!^3 \sum_{l=0}^N \kappa_{4,l}^{1/4}(c). \quad (6.12)$$

Moreover, if $c(\alpha) = 0$ for $|\alpha| = 1$, then

$$|\mathbb{E}(f(S_N(c, \bar{Z}))) - \mathbb{E}(f(G))| \leq 3 \|f\|_\infty N^3 (2N)! N!^3 \sum_{l=0}^N \kappa_{4,l}^{1/2}(c). \quad (6.13)$$

Proof. The proof of (6.13) is an immediate consequence of the results in [17], see (3.38) in Theorem 3.1 and Proposition 3.7 therein. But in order to obtain (6.12) we have to complete the argument from [17]. Since the argument is essentially the same we just sketch the proof and in particular we explain why $\kappa_{4,l}^{1/4}(c)$ appears instead of $\kappa_{4,l}^{1/2}(c)$. Let us briefly recall the notations from [17]. For a symmetric kernel $\phi_n \in L^1(\mathbb{R}_+^n)$ one denotes by $I_n(\phi_n)$ the multiple stochastic integral with kernel ϕ_n . This is an element of the Wiener space and the Malliavin derivative and the Ornstein operator for it are defined as

$$D_s I_n(\phi_n) = n I_{n-1}(\phi_n(\circ, s)), \quad L I_n(\phi_n) = -n I_n(\phi_n). \quad (6.14)$$

Consider now a functional $F_N = \sum_{n=1}^N I_n(\phi_n)$. The operators DF_N and LF_N extend by linearity. Now, (3.38) in [17] says that

$$|\mathbb{E}(f(F_N)) - \mathbb{E}(f(G))| \leq 2 \|f\|_\infty (\mathbb{E}((1 - \langle DF_N, -DL^{-1}F_N \rangle)^2))^{1/2}. \quad (6.15)$$

So our aim now is to estimate the quantity in the right hand side of (6.14). This is done in Proposition 3.7 from [17] but there one considers multiple integrals $I_n(\phi_n)$ with $n \geq 2$ only. If $I_1(\phi_1)$ comes in also, one more term appears and we explain this now. Following [17] we use (6.14) and we obtain

$$\begin{aligned} \langle DF_N, -DL^{-1}F_N \rangle &= n \int_{\mathbb{R}_+} I_{n-1}(\phi_n(\circ, s)) I_{m-1}(\phi_m(\circ, s)) ds \\ &= \sum_{n,m=1}^N n \int_{\mathbb{R}_+} I_{n-1}(\phi_n(\circ, s)) I_{m-1}(\phi_m(\circ, s)) ds \\ &= A + \|\phi_1\|_{L^2(\mathbb{R}_+)}^2 + B + B' \end{aligned}$$

with

$$\begin{aligned} A &= \sum_{n,m=2}^N n \int_{\mathbb{R}_+} I_{n-1}(\phi_n(\circ, s)) I_{m-1}(\phi_m(\circ, s)) ds, \\ B &= \sum_{m=2}^N \int_{\mathbb{R}_+} \phi_1(s) I_{m-1}(\phi_m(\circ, s)) ds = \sum_{m=2}^N I_m(\phi_1 \otimes_1 \phi_m), \\ B' &= \sum_{n=2}^N n \int_{\mathbb{R}_+} I_{n-1}(\phi_n(\circ, s)) \phi_1(s) ds = \sum_{n=2}^N n I_n(\phi_1 \otimes_1 \phi_n). \end{aligned}$$

Using the product formula for multiple stochastic integrals (see (2.29) in [17] for this formula) one obtains

$$\begin{aligned}
A &= \sum_{n,m=2}^N n \sum_{r=0}^{n \wedge m-1} r! \binom{n-1}{r} \binom{m-1}{r} I_{n+m-2-2r} \left(\int_{\mathbb{R}_+} \phi_n(\circ, s) \widetilde{\otimes}_r \phi_m(\circ, s) ds \right) \\
&= \sum_{n,m=2}^N n \sum_{r=0}^{n \wedge m-1} r! \binom{n-1}{r} \binom{m-1}{r} I_{n+m-2-2r} (\phi_n \widetilde{\otimes}_{r+1} \phi_m) \\
&= \sum_{n,m=2}^N n \sum_{r=1}^{n \wedge m} (r-1)! \binom{n-1}{r-1} \binom{m-1}{r-1} I_{n+m-2r} (\phi_n \widetilde{\otimes}_r \phi_m) \\
&= \sum_{n=2}^N n! \|\phi_n\|_{L^2(\mathbb{R}_+^n)}^2 + A'
\end{aligned}$$

with A' just defined by the above equality: so it represents the sum over (n, m, r) such that $(n, m, r) \neq (n, n, n)$. Notice that in this case

$$\begin{aligned}
\|I_{n+m-2r}(\phi_n \widetilde{\otimes}_r \phi_m)\|_2 &= (n+m)! \|\phi_n \widetilde{\otimes}_r \phi_m\|_2 \leq (n+m)! \kappa_{4,n}^{1/2}(c) \vee \kappa_{4,m}^{1/2}(c) \\
&\leq (n+m)! (\kappa_{4,n}^{1/2}(c) + \kappa_{4,m}^{1/2}(c))
\end{aligned}$$

the last inequality being a consequence of well-known facts, which has been here collected in Appendix B, see (B.5) and (B.3). So

$$\|A'\|_2 \leq N^3 (2N)! \times N!^3 \sum_{n=2}^N \kappa_{4,n}^{1/2}(c).$$

And using (B.7) we get $\|I_m(\phi_1 \otimes_1 \phi_m)\|_2 = m! \|\phi_1 \otimes_1 \phi_m\|_2 \leq m! \|\phi_1\|_2 \kappa_{4,n}^{1/4}(c)$ (we stress that $\kappa_{4,n}^{1/4}(c)$ appears here instead of $\kappa_{4,n}^{1/2}(c)$) so that

$$\|B\|_2 + \|B'\|_2 \leq 2N N! \sum_{n=2}^N \kappa_{4,n}^{1/4}(c).$$

We suppose now that $\sum_{n=1}^N n! \|\phi_n\|_{L^2(\mathbb{R}_+^n)}^2 = 1$ and we write $\langle DF_N, -DL^{-1}F_N \rangle = 1 + A' + B + B'$ so that

$$\|1 - \langle DF_N, DL^{-1}F_N \rangle\|_2 \leq \|A'\|_2 + \|B\|_2 + \|B'\|_2 \leq 3N^3 (2N)! \times N!^3 \sum_{n=2}^N \kappa_{4,n}^{1/4}(c)$$

and using (6.15) this gives

$$|\mathbb{E}(f(F_N)) - \mathbb{E}(f(G))| \leq 3 \|f\|_\infty N^3 (2N)! \times N!^3 \left(\sum_{n=2}^N \kappa_{4,n}^{1/4}(c) \right). \quad (6.16)$$

Of course, $\kappa_{4,n}^{1/4}(c)$ can be replaced by $\kappa_{4,n}^{1/2}(c)$ if $\phi_1 = 0$.

We come now back to stochastic series and we define $f_{c(m)}$ to be the kernels which correspond to the coefficients $c(\alpha) : f_{c(m)}(t_1, \dots, t_m) = c(\alpha)$ if $t_i \in [\alpha_i, \alpha_i + 1), i = 1, \dots, m$. Then $F_N = S_N(c, \overline{Z})$ and the hypothesis (6.1) says that $\sum_{n=1}^N n! \|f_{c(n)}\|_{L^2(\mathbb{R}_+^n)}^2 = 1$. And using (6.16) we obtain (6.12). \square

The main result in this section is the following:

Theorem 6.3 *Let $(Z_k)_{k \in \mathbb{N}}$ satisfy (5.4) and such that $Z_k \in \mathcal{L}(r, \varepsilon)$. We also assume that $\sup_k \|Z_k\|_p < \infty$ for every $p \geq 1$ and we suppose that (6.1) and (6.2) hold true. There exist some constants $C, a \geq 1$ such that for every $N \in \mathbb{N}$ and every bounded and measurable function f one has*

$$|\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(G))| \leq C \frac{\|f\|_\infty}{(rm(r))^a} N^3 (2N)! N!^3 (\bar{\kappa}_N(c) + \bar{\delta}_N(c)). \quad (6.17)$$

As a consequence, there exist $C, a \geq 1$ such that for every $N \in \mathbb{N}$ and every bounded and measurable function f one has

$$|\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(G))| \leq C \frac{\|f\|_\infty}{(rm(r))^a} N^3 (2N)! N!^3 (1 + \alpha_N^{-1}(c)) \bar{\kappa}_N(c), \quad (6.18)$$

in which $\alpha_N(c) = \min_{m \leq N} |c|_m 1_{|c|_m > 0}$.

Proof. We take a sequence \bar{Z}_k , $k \in \mathbb{N}$, of standard normal r.v.'s and we write

$$|\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(G))| \leq |\mathbb{E}(f(S_N(c, Z))) - \mathbb{E}(f(S_N(c, \bar{Z})))| + |\mathbb{E}(f(S_N(c, \bar{Z}))) - \mathbb{E}(f(G))|.$$

(6.17) now follows by applying Theorem 6.1 with $p_* = 1$ and Theorem 6.2. Moreover, by using (2.13) one has $\bar{\delta}_N(c) \leq \alpha_N^{-1}(c) \bar{\kappa}_N(c)$, so (6.18) immediately follows from (6.17). \square

6.3 A convergence result for infinite series

We consider a sequence $c^{(n)} = (c^{(n)}(\alpha))_\alpha$ of coefficients and the corresponding infinite series $S_\infty(c^{(n)}, Z)$. Our aim is to give sufficient conditions in order to obtain convergence to the Gaussian law in total variation distance. Here are our hypotheses. First we assume the normalization condition

$$i(c^{(n)}) = \sum_{k=1}^{\infty} k! \sum_{|\alpha|=k} c^{(n)}(\alpha)^2 = 1. \quad (6.19)$$

We also assume that for every $p \geq 1$

$$\sup_n \sum_{k=0}^3 N_k(c^{(n)}, M_p^2) < \infty \quad (6.20)$$

where $N_k(c^{(n)}, M_p^2)$ is defined in (2.4). Moreover we suppose that

$$\limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k \geq N} k \times k! \sum_{|\alpha|=k} |c^{(n)}(\alpha)|^2 = 0. \quad (6.21)$$

Notice that this is analogous to the “uniformity condition” in (3.5), which is used by Hu and Nualart [9] for getting convergence in law for infinite series. Then we have the following convergence result:

Theorem 6.4 *Let $(Z_k)_{k \in \mathbb{N}}$ be a sequence of independent centred random variables with $\mathbb{E}(Z_k^2) = 1$ and which have finite moments of any order. Let $c^{(n)} = (c^{(n)}(\alpha))_\alpha$ be a sequence of coefficients which verify (6.19), (6.20), (6.21) and such that*

$$\lim_{n \rightarrow \infty} \kappa_{4,m}(c^{(n)}) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \delta_m(c^{(n)}) = 0, \quad (6.22)$$

for each $m \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} d_{TV}(S_\infty(c^{(n)}, Z), G) = 0 \quad (6.23)$$

where G is a standard Gaussian random variable and

$$d_{TV}(S_\infty(c^{(n)}, Z), G) := \sup_{\|f\|_\infty \leq 1} |\mathbb{E}(f(S_\infty(c^{(n)}, Z))) - \mathbb{E}(f(G))|. \quad (6.24)$$

Remark 6.5 In view of (2.13), a sufficient condition in order that (6.22) holds is the following:

$$\lim_{n \rightarrow \infty} \kappa_{4,m}(c^{(n)}) \left(1 + \frac{1}{|c^{(n)}|_m^2} \mathbf{1}_{\{|c^{(n)}| > 0\}} \right) = 0, \quad \text{for every } m \in \mathbb{N}.$$

Proof of Theorem 6.4. We set $S^N(c, Z) = S_\infty(c, Z) - S_N(c, Z)$. Then, we have

$$\begin{aligned} \sigma_{S_\infty(c^{(n)}, Z)} - \sigma_{S_N(c^{(n)}, Z)} &= |DS_\infty(c^{(n)}, Z)|^2 - |DS_N(c^{(n)}, Z)|^2 \\ &= \langle DS^N(c^{(n)}, Z), DS_\infty(c^{(n)}, Z) + DS_N(c^{(n)}, Z) \rangle. \end{aligned}$$

So, by Cauchy-Schwartz inequality

$$\mathbb{E}(|\sigma_{S_\infty(c^{(n)}, Z)} - \sigma_{S_N(c^{(n)}, Z)}|) \leq \|DS^N(c^{(n)}, Z)\|_2 (\|DS_\infty(c^{(n)}, Z)\|_2 + \|DS_N(c^{(n)}, Z)\|_2).$$

Setting

$$\varepsilon_N(n) = \sum_{k \geq N} k \times k! \sum_{|\alpha|=k} |c^{(n)}(\alpha)|^2,$$

by Proposition 5.3, we have

$$\mathbb{E}(|\sigma_{S_\infty(c^{(n)}, Z)} - \sigma_{S_N(c^{(n)}, Z)}|) \leq \varepsilon_N(n) \cdot 2\varepsilon_0(n)$$

We take $\eta = \frac{1}{2}m(r)$ and we use (5.15) in order to get

$$\begin{aligned} \mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) &\leq \mathbb{P}(|\sigma_{S_\infty(c^{(n)}, Z)} - \sigma_{S_N(c^{(n)}, Z)}| \geq \eta) + \mathbb{P}(\sigma_{S_N(c^{(n)}, Z)} \leq 2\eta) \\ &\leq \frac{1}{\eta} 2\varepsilon_0(n) \varepsilon_N(n) + C_N(\bar{\kappa}_N(c^{(n)}) + \bar{\delta}_N(c^{(n)})), \end{aligned}$$

where C_N is a constant which depends on N but not on n . Now we use (6.22) and we obtain, for each fixed N

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) &\leq \frac{1}{\eta} \limsup_{n \rightarrow \infty} 2\varepsilon_0(n) \varepsilon_N(n) + C_N \limsup_{n \rightarrow \infty} (\bar{\kappa}_N(c^{(n)}) + \bar{\delta}_N(c)) \\ &= \frac{1}{\eta} \limsup_{n \rightarrow \infty} 2\varepsilon_0(n) \varepsilon_N(n). \end{aligned}$$

Then by (6.21),

$$\limsup_{n \rightarrow \infty} \mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) \leq \frac{2}{\eta} \limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \varepsilon_0(n) \varepsilon_N(n) = 0. \quad (6.25)$$

Now we use the regularization Lemma 4.6: for every $\delta > 0$

$$\begin{aligned} \left| \mathbb{E}(f(S_\infty(c^{(n)}, Z))) - \mathbb{E}(f_\delta(S_\infty(c^{(n)}, Z))) \right| &\leq C \|f\|_\infty \left(\mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) + \frac{\sqrt{\delta}}{\eta^p} (1 + \|S(c^{(n)}, Z)\|_{3,p})^a \right) \\ &\leq C \|f\|_\infty \left(\mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) + \frac{\sqrt{\delta}}{\eta^p} C \right) \end{aligned}$$

the last inequality being a consequence of (5.9) and (6.20). And a similar inequality holds for G . So

$$\begin{aligned} \left| \mathbb{E}(f(S_\infty(c^{(n)}, Z))) - \mathbb{E}(f(G)) \right| &\leq \left| \mathbb{E}(f_\delta(S_\infty(c^{(n)}, Z))) - \mathbb{E}(f_\delta(G)) \right| \\ &\quad + C \|f\|_\infty \left(\mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) + \mathbb{P}(\sigma_G \leq \eta) + \frac{\sqrt{\delta}}{\eta^p} C \right) \\ &\leq \left| \mathbb{E}(f_\delta(S_N(c^{(n)}, Z))) - \mathbb{E}(f_\delta(G)) \right| \\ &\quad + \left| \mathbb{E}(f_\delta(S_\infty(c^{(n)}, Z))) - \mathbb{E}(f_\delta(S_N(c^{(n)}, Z))) \right| \\ &\quad + C \|f\|_\infty \left(\mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) + \frac{\sqrt{\delta}}{\eta^p} C \right) \\ &=: A_N^\delta(n) + B_N^\delta(n) + C \|f\|_\infty \left(\mathbb{P}(\sigma_{S_\infty(c^{(n)}, Z)} \leq \eta) + \frac{\sqrt{\delta}}{\eta^p} C \right) \end{aligned}$$

in which we have used the fact that $\sigma_G = 1 > \eta$, so that $\mathbb{P}(\sigma_G \leq \eta) = 0$. Now, by using Theorem 6.3 and by recalling that $\|f_\delta\|_\infty \leq \|f\|_\infty$,

$$\limsup_{n \rightarrow \infty} \sup_{\|f\|_\infty \leq 1} A_N^\delta(n) = 0,$$

for every fixed $N > 0$. Moreover, since $\|f'_\delta\|_\infty \leq \|f\|_\infty/\delta$, we can write

$$B_N^\delta(n) \leq \frac{\|f\|_\infty}{\delta} \mathbb{E}(|S^N(c^{(n)}, Z)|) \leq \frac{\|f\|_\infty}{\delta} \varepsilon_N(n)^{1/2}$$

so that

$$\limsup_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{\|f\|_\infty \leq 1} B_N^\delta(n) = 0$$

because of (6.21), for every $\delta > 0$. By using (6.25) we finally get

$$\limsup_{n \rightarrow \infty} \sup_{\|f\|_\infty \leq 1} \left| \mathbb{E}(f(S_\infty(c^{(n)}, Z))) - \mathbb{E}(f(G)) \right| \leq C \frac{\sqrt{\delta}}{\eta^p}.$$

Since $\delta > 0$ is arbitrary the proof is complete. \square

A Burkholder inequality for Hilbert valued discrete time martingales

We consider a Hilbert space \mathcal{U} and we denote $\|\cdot\|_{\mathcal{U}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ respectively the norm and the inner product on \mathcal{U} . Recall $L_{\mathcal{U}}^p$ and $\mathbb{D}_{\mathcal{U}}^{q,p}$ defined at the beginning of Section 5.

We consider a martingale $M_n \in \mathcal{U}, n \in \mathbb{N}$ and we recall Burkholder's inequality in this framework: for each $p \geq 2$ there exists a universal constant $b_p \geq 1$ such that

$$\|M_n\|_{\mathcal{U},p} \leq b_p \left(\mathbb{E} \left(\left(\sum_{k=1}^n |\Delta_k|_{\mathcal{U}}^2 \right)^{p/2} \right) \right)^{1/p}, \quad \Delta_k = M_k - M_{k-1}. \quad (\text{A.1})$$

As an immediate consequence

$$\|M_n\|_{\mathcal{U},p} \leq b_p \left(\sum_{k=1}^n \|\Delta_k\|_{\mathcal{U},p}^2 \right)^{1/2}. \quad (\text{A.2})$$

Indeed, by using (A.1),

$$\|M_n\|_{\mathcal{U},p}^2 \leq b_p^2 \left\| \sum_{k=1}^n |\Delta_k|_{\mathcal{U}}^2 \right\|_{p/2} \leq b_p^2 \sum_{k=1}^n \|\Delta_k\|_{\mathcal{U},p/2}^2 = b_p^2 \sum_{k=1}^n \|\Delta_k\|_{\mathcal{U},p}^2.$$

We give now estimates which are used in order to upper bound the Sobolev norms of $LS(c, Z)$. Recall the definition of the space $\mathbb{D}_{\mathcal{U}}^{q,p}$ given in Section 5 and we set $\mathbb{D}_{\mathcal{U}}^\infty = \cap_{p \geq 1} \cap_{q \geq 0} \mathbb{D}_{\mathcal{U}}^{q,p}$.

Proposition A.1 *Suppose that $Z_k \in \mathcal{L}(z_k, r, \varepsilon)$, $k \in \mathbb{N}$. Let $B_k, \Lambda_k \in \mathcal{U}$ be random variables such that $B_k, \Lambda_k \in \mathbb{D}_{\mathcal{U}}^\infty$ for every k and B_k is $\sigma(Z_1, \dots, Z_k)$ measurable. Consider the process*

$$Y_m = \sum_{k=1}^{m-1} B_k L Z_{k+1} + \Lambda_m. \quad (\text{A.3})$$

Then for every $q \in \mathbb{N}$ and $p \geq 2$ there exists a universal constant $C \geq 1$ such that

$$\max_{m \leq n} \|Y_m\|_{\mathcal{U},q,p} \leq \frac{C}{r^{q+1}} \times C_{q,p}(B, \Lambda) \quad (\text{A.4})$$

with

$$C_{q,p}(B, \Lambda) = \left(\sum_{k=1}^n \|B_k\|_{\mathcal{U},q,p}^2 \right)^{1/2} + \max_{m \leq n} \|\Lambda_m\|_{\mathcal{U},q,p}. \quad (\text{A.5})$$

Proof. We will use the following facts, proved in Lemma 3.2 in [1]: $\mathbb{E}(LZ_k) = 0$ and there exists a universal constant C such that

$$\|LZ_k\|_{q,p} \leq \frac{C}{r^{q+1}}. \quad (\text{A.6})$$

Step 1. Let $q = 0$, so that $\|Y_m\|_{\mathcal{U},q,p} = \|Y_m\|_{\mathcal{U},p}$. We have to check that

$$\max_{m \leq n} \|Y_m\|_{\mathcal{U},p} \leq \frac{C}{r} \times C_{0,p}(B, \Lambda). \quad (\text{A.7})$$

Since B_k is $\sigma(Z_1, \dots, Z_k)$ measurable and $\mathbb{E}(LZ_{k+1}) = 0$, it follows that $M_m = \sum_{k=1}^{m-1} B_k LZ_{k+1}$ is a martingale. By (A.2)

$$\|M_m\|_{\mathcal{U},p} \leq b_p \left(\sum_{k=1}^m \|LZ_{k+1} B_k\|_{\mathcal{U},p}^2 \right)^{1/2}.$$

Since LZ_{k+1} and B_k are independent,

$$\|LZ_{k+1} B_k\|_{\mathcal{U},p}^2 = \|LZ_{k+1}\|_p^2 \|B_k\|_{\mathcal{U},p}^2 \leq \frac{C}{r^2} \|B_k\|_{\mathcal{U},p}^2.$$

From $Y_m = M_m + \Lambda_m$, we conclude that

$$\|Y_m\|_{\mathcal{U},p} \leq \|M_m\|_{\mathcal{U},p} + \|\Lambda_m\|_{\mathcal{U},p} \leq \frac{C}{r} \left(\left(\sum_{k=1}^m \|B_k\|_{\mathcal{U},p}^2 \right)^{1/2} + \|\Lambda_m\|_{\mathcal{U},p} \right)$$

and the statement holds for $q = 0$.

Step 2. We estimate the derivatives of Y_m . We have

$$\bar{Y}_m := DY_m = \sum_{k=1}^{m-1} \bar{B}_k LZ_{k+1} + \bar{\Lambda}_m.$$

with $\bar{B}_k = DB_k$ and $\bar{\Lambda}_m = \sum_{k=1}^{m-1} DLZ_{k+1} B_k + D\Lambda_m$. Notice that \bar{Y}_m , \bar{B}_k and $\bar{\Lambda}_m$ take values in $\mathcal{H}(\mathcal{U})$ (defined in (5.1)). So, by applying the step above, we get

$$\max_{m \leq n} \|DY_m\|_{\mathcal{H}(\mathcal{U}),p} \leq \frac{C}{r} C_{0,p}(\bar{B}, \bar{\Lambda}),$$

where

$$C_{0,p}(\bar{B}, \bar{\Lambda}) = \max_{m \leq n} \left(\left(\sum_{k=1}^m \|\bar{B}_k\|_{\mathcal{H}(\mathcal{U}),p}^2 \right)^{1/2} + \|\bar{\Lambda}_m\|_{\mathcal{H}(\mathcal{U}),p} \right).$$

If we prove that

$$C_{0,p}(\bar{B}, \bar{\Lambda}) \leq \frac{C}{r} \times C_{1,p}(B, \Lambda) \quad (\text{A.8})$$

(hereafter, $C > 0$ denotes a constant that may vary) and recalling that $C_{0,p}(B, \Lambda) \leq C_{1,p}(B, \Lambda)$, then we obtain

$$\max_{m \leq n} \|Y_m\|_{\mathcal{U},1,p} \leq \frac{C}{r^2} C_{1,p}(B, \Lambda).$$

And by iteration, we get (A.4). So, let us prove (A.8).

We have $\|\bar{B}_k\|_{\mathcal{H}(\mathcal{U}),p} = \|DB_k\|_{\mathcal{H}(\mathcal{U}),p} \leq \|B_k\|_{\mathcal{U},1,p}$. We analyze now $\bar{\Lambda}_m$. First, $\|D\Lambda_k\|_{\mathcal{H}(\mathcal{U}),p} \leq \|\Lambda_m\|_{\mathcal{U},1,p}$. Let $I_m := \sum_{k=1}^{m-1} DLZ_{k+1} B_k$. Since $D_p LZ_{k+1} = 0$ if $p \neq k+1$ we obtain

$$|I_m|_{\mathcal{H}(\mathcal{U})}^2 = \sum_{k=1}^{m-1} |D_{k+1} LZ_{k+1}|^2 |B_k|_{\mathcal{U}}^2.$$

Recalling that $D_{k+1}LZ_{k+1}$ and B_k are independent and that $\|D_{k+1}LZ_{k+1}\|_p^2 \leq Cr^{-2}$, we can write

$$\begin{aligned} \|I_m\|_{\mathcal{H}(\mathcal{U}),p} &= \|I_m\|_{\mathcal{H}(\mathcal{U})}^2 \|p/2\|^{1/2} \leq \left(\sum_{k=1}^{m-1} \| |D_{k+1}LZ_{k+1}|^2 |B_k|_{\mathcal{U}}^2 \|_{p/2} \right)^{1/2} \\ &= \left(\sum_{k=1}^{m-1} \|D_{k+1}LZ_{k+1}\|_p^2 \|B_k\|_{\mathcal{U},p}^2 \right)^{1/2} \leq \frac{C}{r} \times \left(\sum_{k=1}^{m-1} \|B_k\|_{\mathcal{U},p}^2 \right)^{1/2}. \end{aligned}$$

By inserting all these estimates, we get (A.8). \square

B The L^p estimates in Lemma 5.6

B.1 Contractions and cumulants

We briefly recall some well known facts concerning contractions of kernels and cumulants, and we give some easy consequences which are used in our paper. The results in this section involve $|c|_m$ (see (2.3)), $\delta_m(c)$ (see (2.5)) and $\kappa_{4,m}(c)$ (see (2.10)). We denote by $c_{(m)}(\alpha) = 1_{\{|\alpha|=m\}}c(\alpha)$, so $c_{(m)} \in \mathcal{H}^{\otimes m}$ represents the restriction of c to Γ_m . Then for $m, n \in \mathbb{N}$ and $0 \leq r \leq m \wedge n$ we define the contraction $c_{(m)} \otimes_r c_{(n)} \in \mathcal{H}^{\otimes(m+n-2r)}$ as follows

$$c_{(m)} \otimes_r c_{(n)}(\alpha, \beta) = \sum_{|\gamma|=r} c_{(m)}((\alpha, \gamma)) c_{(n)}((\beta, \gamma)) = \sum_{|\gamma|=r} c((\alpha, \gamma)) c((\beta, \gamma)) \quad (\text{B.1})$$

where $\alpha = (\alpha_1, \dots, \alpha_{m-r}), \beta = (\beta_1, \dots, \beta_{n-r})$. Since for $m \neq n$, $c_{(m)} \otimes_r c_{(n)}$ is not symmetric, we define $c_{(m)} \widetilde{\otimes}_r c_{(n)}$ to be the symmetrization of $c_{(m)} \otimes_r c_{(n)}$: for $\eta \in \Gamma_{m+n-2r}$,

$$c_{(m)} \widetilde{\otimes}_r c_{(n)}(\eta) = \frac{1}{(n+m-2r)!} \sum_{\pi \in \Pi_{m+n-2r}} c_{(m)} \otimes_r c_{(n)}(\mathbf{p}_{m-r}(\eta_\pi), \mathbf{q}_{n-r}(\eta_\pi)), \quad (\text{B.2})$$

in which Π_{m+n-2r} denotes the permutations of $\{1, \dots, m+n-2r\}$, for $\pi \in \Pi_{m+n-2r}$ then $\eta_\pi = (\eta_{\pi_1}, \dots, \eta_{\pi_{m+n-2r}})$, \mathbf{p}_{m-r} is the projection on the first $m-r$ coordinates and \mathbf{q}_{n-r} is the projection on the last $n-r$ coordinates, with the convention $\mathbf{p}_0 = \mathbf{q}_0 = \emptyset$.

Finally we recall Remark 2.1: that if $Z = (Z_k)_{k \in \mathbb{N}}$ with Z_k independent standard normal random variables then $\Phi_m(c, Z) = I_m(f_{c_{(m)}})$ is the multiple stochastic integral with the piecewise constant kernel $f_{c_{(m)}}(t_1, \dots, t_m) = c(\alpha)$ if $t_i \in [\alpha_i, \alpha_i + 1), i = 1, \dots, m$. So we come back to the Wiener space (the results known in the literature usually concern multiple stochastic integrals) and we summarize all the needed results in the next lemma.

Lemma B.1 • *One has*

$$\kappa_{4,m}(c) = \sum_{r=1}^{m-1} m!^2 \binom{m}{r}^2 \{ \|c_{(m)} \otimes_r c_{(m)}\|^2 + \binom{2m-2r}{m-r} \|c_{(m)} \widetilde{\otimes}_r c_{(m)}\|^2 \}. \quad (\text{B.3})$$

• *For $0 \leq r \leq m \wedge n$, one has*

$$\|c_{(m)} \widetilde{\otimes}_r c_{(n)}\|^2 \leq \frac{1}{2} (\|c_{(m)} \otimes_{m-r} c_{(m)}\|^2 + \|c_{(n)} \otimes_{n-r} c_{(n)}\|^2) \quad (\text{B.4})$$

$$\|c_{(m)} \otimes_r c_{(n)}\|^2 \leq \|c_{(m)} \otimes_{m-r} c_{(m)}\| \|c_{(n)} \otimes_{n-r} c_{(n)}\| \quad (\text{B.5})$$

- For $0 < r < m \wedge n$

$$\begin{aligned} \|c_{(m)} \tilde{\otimes}_r c_{(n)}\| &\leq \max \left(\frac{\sqrt{\kappa_{4,m}(c)}}{m! \binom{m}{r}}, \frac{\sqrt{\kappa_{4,n}(c)}}{n! \binom{n}{r}} \right) \\ \|c_{(m)} \otimes_r c_{(n)}\| &\leq \max \left(\frac{\sqrt{\kappa_{4,m}(c)}}{m! \binom{m}{r}}, \frac{\sqrt{\kappa_{4,n}(c)}}{n! \binom{n}{r}} \right) \end{aligned} \quad (\text{B.6})$$

- For $1 \leq m \leq n-1$

$$\|c_{(m)} \otimes_m c_{(n)}\| \leq \|c_{(m)}\|^2 \|c_{(n)} \otimes_{n-m} c_{(n)}\| \leq \|c_{(m)}\|^2 \left(\frac{\sqrt{\kappa_{4,n}(c)}}{n! \binom{n}{n-m}} \right)^{1/2}. \quad (\text{B.7})$$

- The following estimate for the influence factor $\delta_m(c)$ holds:

$$\delta_m(c) \leq \frac{1}{|c|_m} |c_{(m)} \otimes_{m-1} c_{(m)}|_{2m-2} \leq \frac{\sqrt{\kappa_{4,m}(c)}}{m!m |c|_m} \quad (\text{B.8})$$

Proof. We first recall that $\kappa_{4,m}(c) = \kappa_4(\Phi_m(c, Z))$ with $Z_k, k \in \mathbb{N}$, standard normal. So, the identity (B.3) is proved in [23] for iterated integrals and remains true for stochastic series because $\|f_{c_{(m)}}\|_{L(\mathbb{R}_+^m)} = \|c_{(m)}\|$ and $f_{c_{(m)}} \otimes f_{c_{(n)}} = f_{c_{(m)} \otimes c_{(n)}}$. (B.4) is straightforward (but see also formula (13) in [22]) and (B.5) appears in [22] and [19]. (B.6) is an immediate consequence of (B.4)-(B.5) and (B.3). Concerning (B.7), straightforward computations give

$$\|c_{(m)} \otimes_m c_{(n)}\|^2 = \sum_{\rho, \bar{\rho} \in \Gamma_m} c_{(m)}(\rho) c_{(m)}(\bar{\rho}) c_{(n)} \otimes_{n-m} c_{(n)}(\rho, \bar{\rho}).$$

By using the Cauchy-Schwarz inequality we get $\|c_{(m)} \otimes_m c_{(n)}\|^2 \leq \|c_{(m)}\|^2 \|c_{(n)} \otimes_{n-m} c_{(n)}\|$. Last inequality in (B.7) follows from (B.6). Finally, the inequality (B.8) has been proved in [18] \square

B.2 Some L^p estimates for series

B.2.1 The basic lemma

We start with the basic definitions of this section.

Assumption B.2 1. We fix $m, n \geq 0$ integers and we consider a coefficient

$$a : \Gamma_m \times \Gamma_n \rightarrow \mathbb{R}, \quad (\alpha, \beta) \mapsto a(\alpha, \beta)$$

that satisfies:

- for $n + m \geq 1$, as a function of $\gamma = (\alpha, \beta) \in \Gamma_{m+n}$, a is null on the diagonals;
- for $m, n \geq 2$, $\Gamma_m \ni \alpha \mapsto a(\alpha, \beta)$ and $\Gamma_n \ni \beta \mapsto a(\alpha, \beta)$ are both symmetric (so a is symmetric in each argument, separately).

We define

$$|a|_{m,n,J} = \left(\sum_{\alpha \in \Gamma_m(J)} \sum_{\beta \in \Gamma_n(J)} a^2(\alpha, \beta) \right)^{1/2} \quad (\text{B.9})$$

2. Take now $\bar{a} = (a_j)_{j \in \mathbb{N}}$, with $a_j : \Gamma_m \times \Gamma_n \rightarrow \mathbb{R}$ which satisfies the hypotheses in 1. and furthermore with

$$a_j(\gamma) = 0 \text{ if } j \in \gamma.$$

In this case, we denote

$$|\bar{a}|_{m,n,J} = \left(\sum_{j=1}^{\infty} \sum_{\alpha \in \Gamma_m(J)} \sum_{\beta \in \Gamma_n(J)} a_j^2(\alpha, \beta) \right)^{1/2}. \quad (\text{B.10})$$

3. We consider a sequence of centred and independent random variables $(Z_k, Y_k, \tilde{\chi}_k)$, $k \in \mathbb{N}$ (with $\tilde{\chi}_j = \chi_j - \mathbb{E}(\chi_j)$, Z_k and χ_k having the usual meaning) and we denote

$$M_p(Z, Y) = \sup_k \|Z_k\|_p \vee \|Y_k\|_p. \quad (\text{B.11})$$

Notice that we do not require that Y_k is independent of Z_k and/or of $\tilde{\chi}_k$. We are interested in the following double series: for a fixed a as in 1.,

$$t_{m,n}(J, a) = \sum_{\alpha \in \Gamma_m(J)} \sum_{\beta \in \Gamma_n(J)} Z^\alpha Y^\beta a(\alpha, \beta) \quad (\text{B.12})$$

and for $\bar{a} = (a_j)_{j \in \mathbb{N}}$ as in 2.,

$$T_{m,n}(J, \bar{a}) = \sum_{\alpha \in \Gamma_m(J)} \sum_{\beta \in \Gamma_n(J)} Z^\alpha Y^\beta \sum_{j=1}^{\infty} a_j(\alpha, \beta) \tilde{\chi}_j. \quad (\text{B.13})$$

Lemma B.3 Under Assumption B.2 one has

$$\|t_{m,n}(J, a)\|_p \leq ((m+n)!)^{1/2} (\sqrt{2} b_p M_p(Z, Y))^{(n+m)} |a|_{m,n,J}, \quad (\text{B.14})$$

$$\|T_{m,n}(J, \bar{a})\|_p \leq \left(\frac{8b_p^2(4^{m+n} - 1)(m+n)!}{3} \right)^{1/2} (\sqrt{2} b_p M_p(Z, Y))^{m+n} |\bar{a}|_{m,n,J}. \quad (\text{B.15})$$

Proof. We recall that a and a_j are all null on the diagonals. So, the sums in (B.12) and (B.13) are really done on the multi-indexes α and β that do not have common components. So, we consider such kind of indexes.

Step 1. We denote

$$\begin{aligned} \Lambda'_{m,n}(J') &= (\Gamma_m(J') \setminus \Gamma_m(J' - 1)) \times \Gamma_n(J' - 1) \\ \Lambda''_{m,n}(J') &= \Gamma_m(J') \times (\Gamma_n(J') \setminus \Gamma_n(J' - 1)). \end{aligned}$$

So $(\alpha, \beta) \in \Lambda'_{m,n}(J')$ if $\max_{j=1, \dots, m} \alpha_j = J'$ and $\max_{j=1, \dots, n} \beta_j \leq J' - 1$. And the definition of $\Lambda''_{m,n}(J')$ is similar, with α replaced by β . Finally we put

$$\Lambda_{m,n}(J') = \Lambda'_{m,n}(J') \cup \Lambda''_{m,n}(J').$$

So, we have

$$t_{m,n}(J, a) = \sum_{\alpha \in \Gamma_m(J)} \sum_{\beta \in \Gamma_n(J)} Z^\alpha Y^\beta a(\alpha, \beta) = \sum_{J'=1}^J \sum_{(\alpha, \beta) \in \Lambda_{m,n}(J')} Z^\alpha Y^\beta a(\alpha, \beta).$$

In order to prove (B.14), the first step is to establish a recurrence formula. We define

$$(Q'_J a)(\alpha, \beta) = a((\alpha, J), \beta) \quad \text{and} \quad (Q''_J a)(\alpha, \beta) = a(\alpha, (\beta, J)) \quad (\text{B.16})$$

and we write

$$t_{m,n}(J, a) = \sum_{J'=1}^J \sum_{(\alpha, \beta) \in \Lambda'_{m,n}(J')} Z^\alpha Y^\beta a(\alpha, \beta) + \sum_{J'=1}^J \sum_{(\alpha, \beta) \in \Lambda''_{m,n}(J')} Z^\alpha Y^\beta a(\alpha, \beta).$$

But $(\alpha, \beta) \in \Lambda'_{m,n}(J')$ if and only if $\beta \in \Gamma_n(J' - 1)$ and α contains J' , the remaining entries forming a multi-index in $\Gamma_{m-1}(J' - 1)$. And similarly for $(\alpha, \beta) \in \Lambda''_{m,n}(J')$, changing the role to α and β . So, by using the symmetry of $\alpha \mapsto a(\alpha, \beta)$ and $\beta \mapsto a(\alpha, \beta)$, we can write

$$\begin{aligned} t_{m,n}(J, a) &= m \sum_{J'=1}^J Z_{J'} \sum_{\alpha \in \Gamma_{m-1}(J'-1)} \sum_{\beta \in \Gamma_n(J'-1)} Z^\alpha Y^\beta a((\alpha, J'), \beta) \\ &\quad + n \sum_{J'=1}^J Y_{J'} \sum_{\alpha \in \Gamma_m(J'-1)} \sum_{\beta \in \Gamma_{n-1}(J'-1)} Z^\alpha Y^\beta a(\alpha, (\beta, J')) \\ &= \sum_{J'=1}^J (Z_{J'} m t_{m-1,n}(J' - 1, Q'_{J'} a) + Y_{J'} n t_{m,n-1}(J' - 1, Q''_{J'} a)). \end{aligned}$$

Let $\mathcal{G}_n = \sigma\{(Z_k, Y_k, \tilde{\chi}_k) : k \leq n\}$. Notice that $t_{m-1,n}(J' - 1, Q'_{J'} a)$ and $t_{m,n-1}(J' - 1, Q''_{J'} a)$ are $\mathcal{G}_{J'-1}$ measurable so the above sums are martingales with respect to the filtration \mathcal{G}_n , $n \in \mathbb{N}$, and we may use Burkholder's inequality. Using the above recurrence formula and the recurrence hypotheses (B.14) we obtain

$$\begin{aligned} \|t_{m,n}(J, a)\|_p^2 &\leq (\sqrt{2}b_p M_p(Z, Y))^2 \left(\sum_{J'=1}^J m^2 \|t_{m-1,n}(J', Q'_{J'} a)\|_p^2 + \sum_{J'=1}^{J-1} n^2 \|t_{m,n-1}(J', Q''_{J'} a)\|_p^2 \right) \\ &\leq (m+n-1)! (\sqrt{2}b_p M_p(Z, Y))^{2(n+m)} \left(\sum_{J'=1}^J \sum_{\alpha \in \Gamma_{m-1}(J'-1)} \sum_{\beta \in \Gamma_n(J'-1)} m^2 a^2((\alpha, J'), \beta) \right. \\ &\quad \left. + \sum_{J'=1}^J \sum_{\alpha \in \Gamma_m(J'-1)} \sum_{\beta \in \Gamma_{n-1}(J'-1)} n^2 a^2(\alpha, (\beta, J')) \right) \\ &\leq (m+n-1)! (\sqrt{2}b_p M_p(Z, Y))^{2(n+m)} \times (m+n) \sum_{\alpha \in \Gamma_m(J)} \sum_{\beta \in \Gamma_n(J)} a^2(\alpha, \beta), \end{aligned}$$

so (B.14) is proved.

Step 2. We prove (B.15). We write

$$\begin{aligned} T_{m,n}(J, \bar{a}) &= \sum_{J'=1}^J \sum_{(\alpha, \beta) \in \Lambda_{m,n}(J')} Z^\alpha Y^\beta \sum_{j=J'+1}^\infty a_j(\alpha, \beta) \tilde{\chi}_j + \sum_{J'=1}^J \sum_{(\alpha, \beta) \in \Lambda_{m,n}(J')} Z^\alpha Y^\beta \sum_{j=1}^{J'-1} a_j(\alpha, \beta) \tilde{\chi}_j \\ &= A_{m,n}(J, \bar{a}) + B_{m,n}(J, \bar{a}). \end{aligned}$$

Notice that the term $j = J'$ does not appear: for $(\alpha, \beta) \in \Lambda_{m,n}(J')$ then $J' \in (\alpha, \beta)$, so $a_{J'}(\alpha, \beta) = 0$ by our assumption. Now we write

$$A_{m,n}(J, \bar{a}) = \sum_{j=2}^\infty \tilde{\chi}_j \sum_{J'=1}^{J \wedge (j-1)} \sum_{(\alpha, \beta) \in \Lambda_{m,n}(J')} Z^\alpha Y^\beta a_j(\alpha, \beta) = \sum_{j=1}^\infty \tilde{\chi}_j t_{m,n}(J \wedge (j-1), a_j).$$

We use Burkholder's inequality and (B.14) in order to obtain

$$\begin{aligned}
\|A_{m,n}(J, \bar{a})\|_p &\leq 2b_p \left(\sum_{j=2}^{\infty} \|t_{m,n}(J \wedge (j-1), a_j)\|_p^2 \right)^{1/2} \\
&\leq 2b_p (\sqrt{2}b_p M_p(Z, Y))^{(n+m)} ((m+n)!)^{1/2} \times \\
&\quad \times \left(\sum_{j=1}^{\infty} \sum_{\alpha \in \Gamma_m(J \wedge (j-1))} \sum_{\beta \in \Gamma_n(J \wedge (j-1))} a_j^2(\alpha, \beta) \right)^{1/2}.
\end{aligned} \tag{B.17}$$

Step 3. We estimate now $\|B_{m,n}(J, \bar{a})\|_p$. We write

$$\begin{aligned}
B_{m,n}(J, \bar{a}) &= \sum_{J'=1}^J Z_{J'm} \sum_{\alpha \in \Gamma_{m-1}(J'-1)} \sum_{\beta \in \Gamma_n(J'-1)} Z^\alpha Y^\beta \sum_{j=1}^{\infty} 1_{\{j < J'\}}(Q'_{J'} a_j)(\alpha, \beta) \tilde{\chi}_j \\
&\quad + \sum_{J'=1}^J Y_{J'n} \sum_{\alpha \in \Gamma_m(J'-1)} \sum_{\beta \in \Gamma_{n-1}(J'-1)} Z^\alpha Y^\beta \sum_{j=1}^{\infty} 1_{\{j < J'\}}(Q''_{J'} a_j)(\alpha, \beta) \tilde{\chi}_j \\
&= \sum_{J'=1}^J (Z_{J'm} T_{m-1,n}(J'-1, q'_{J'} \bar{a}) + Y_{J'n} T_{m,n-1}(J'-1, q''_{J'} \bar{a})).
\end{aligned}$$

with

$$\begin{aligned}
(q'_{J'} \bar{a})_j(\alpha, \beta) &= 1_{\{j < J'\}}(Q'_{J'} a_j)(\alpha, \beta) = 1_{\{j < J'\}} a_j((\alpha, J'), \beta) \\
(q''_{J'} \bar{a})_j(\alpha, \beta) &= 1_{\{j < J'\}}(Q''_{J'} a_j)(\alpha, \beta) = 1_{\{j < J'\}} a_j(\alpha, (\beta, J')),
\end{aligned}$$

$Q'_{J'} a_j$ and $Q''_{J'} a_j$ being defined in (B.16). By using Burkholder's inequality,

$$\|B_{m,n}(J, \bar{a})\|_p^2 \leq 2b_p^2 M_p^2(Z, Y) \sum_{J'=1}^J (m^2 \|T_{m-1,n}(J'-1, q'_{J'} \bar{a})\|_p^2 + n^2 \|T_{m,n-1}(J'-1, q''_{J'} \bar{a})\|_p^2).$$

So finally

$$\begin{aligned}
\|T_{m,n}(J, \bar{a})\|_p^2 &\leq 2 \|A_{m,n}(J, \bar{a})\|_p^2 + 2 \|B_{m,n}(J, \bar{a})\|_p^2 \\
&\leq 2 \|A_{m,n}(J, \bar{a})\|_p^2 + 2b_p^2 M_p^2(Z, Y) \sum_{J'=1}^J (m^2 \|T_{m-1,n}(J'-1, q'_{J'} \bar{a})\|_p^2 + n^2 \|T_{m,n-1}(J'-1, q''_{J'} \bar{a})\|_p^2)
\end{aligned}$$

Using the recurrence hypothesis and (B.14) we conclude the proof of (B.15). \square

B.2.2 The “product formula”

We have to deal with $|S(f, Z)|^2$ with

$$S(f, Z) = \sum_{m \geq 0} \sum_{\alpha \in \Gamma_m} f_{(m)}(\alpha) Z^\alpha,$$

where $f = \{f_{(m)}\}_m$ is a symmetric sequence of coefficients in $\mathcal{H}^{\otimes m}$, $m \in \mathbb{N}$, which are null on all diagonals. Note that the case $m = 0$ is allowed, by setting $f_{(0)} \in \mathbb{R}$, $\Gamma_0 = \{\emptyset\}$ and $Z^\emptyset = 1$. We then study $|S(f, Z)|^2$. To this purpose, we recall that Π_n denotes the set of all permutations of $(1, \dots, n)$; for $\eta \in \Gamma_n$ and $\pi \in \Pi_n$, we set $\eta_\pi = (\eta_{\pi_1}, \dots, \eta_{\pi_n})$.

Lemma B.4 *We have*

$$|S(f, Z)|^2 = \sum_{m \geq 0} \sum_{n \geq 0} \sum_{\gamma \in \Gamma_m} \sum_{\eta \in \Gamma_n} (Z^\gamma)^2 Z^\eta A_{n,m}[f](\eta, \gamma) \quad (\text{B.18})$$

where, for $n, m \geq 0$, $\eta \in \Gamma_n$ and $\gamma \in \Gamma_m$,

$$A_{n,m}[f](\eta, \gamma) = \frac{m!}{n!} \sum_{a=0}^n \binom{a+m}{m} \binom{n-a+m}{m} \sum_{\pi \in \Pi_n} f_{(a+m)}(\mathbf{p}_a(\eta_\pi), \gamma) f_{(n-a+m)}(\mathbf{q}_{n-a}(\eta_\pi), \gamma) \quad (\text{B.19})$$

in which, for $\eta \in \Gamma_n$ and $a = 0, 1, \dots, n$,

$$\mathbf{p}_a(\eta) = \begin{cases} \emptyset & \text{for } a = 0 \\ (\eta_1, \dots, \eta_a) & \text{for } 1 \leq a \leq n \end{cases} \quad \text{and} \quad \mathbf{q}_{n-a}(\eta) = \begin{cases} (\eta_{a+1}, \dots, \eta_n) & \text{for } 0 \leq a \leq n-1 \\ \emptyset & \text{for } a = n \end{cases} \quad (\text{B.20})$$

Note that $A_{m,n}[f](\eta, \gamma) = 0$ if η and γ have common components and the maps $\eta \mapsto A_{m,n}[f](\eta, \gamma)$ and $\gamma \mapsto A_{m,n}[f](\eta, \gamma)$ are both symmetric.

Remark B.5 *People working in Wiener chaos use the product formula for multiple stochastic integrals in order to compute $|S(f, Z)|^2$. But this is not possible here. Suppose that we want to do it in the case where the Z_k 's are standard normal - so $S(f, Z)$ is a sum of multiple stochastic integrals (Remark 2.1). We stress that the kernels of these integrals are piecewise constant on the intervals $[k, k+1)$ and if we use the product formula we get multiple stochastic integrals with kernels which are no more piecewise constant on the same grid $[k, k+1)$, $k \in \mathbb{N}$ - so we get out from our framework. Put it otherwise: stochastic series with Gaussian random variables Z_k , $k \in \mathbb{N}$, are functionals of the increments of the Brownian motion on $[k, k+1)$, $k \in \mathbb{N}$. And if we use the product formula for such series we obtain functionals of the whole Brownian path. Just as an example, if W_t is a Brownian motion and if $Z_1 = W_1$ then $Z_1^2 = 2 \int_0^1 W_s dW_s + 1$. Moreover, in the general case we have no Itô formula which permits to get the above representation of Z_1^2 . So we have to replace the product formula by (B.18). This leads to some algebraic difficulties but not only. In fact, the product formula allows to eliminate squares - one comes down to linear combinations of multiple stochastic integrals and this is very nice because then one may use the standard Burkholder inequality for them. But here this does not work and then we have to estimate double series as the ones defined in (B.12) and (B.13).*

Proof of Lemma B.4. For α, β multi-indexes, let $\#\alpha \cap \beta$ denote the number of the components which are common to both α and β . For $m, n \geq 0$ and $r = 0, \dots, m \wedge n$, we set $\Lambda_r^{m,n} = \{(\alpha, \beta) \in \Gamma_m \times \Gamma_n : \#\alpha \cap \beta = r\}$. Then,

$$S(f, Z) = \sum_{m \geq 0} \sum_{n \geq 0} \sum_{(\alpha, \beta) \in \Gamma_m \times \Gamma_n} f_{(m)}(\alpha) f_{(n)}(\beta) Z^\alpha Z^\beta = \sum_{m \geq 0} \sum_{n \geq 0} \sum_{r=0}^{m \wedge n} \sum_{(\alpha, \beta) \in \Lambda_r^{m,n}} f_{(m)}(\alpha) f_{(n)}(\beta) Z^\alpha Z^\beta.$$

We set $\tilde{\Gamma}_m$ as the set of the non-ordered multi-index, that is the set of all subsets of \mathbb{N}^m , and Π_m the set of all permutations of $(1, \dots, m)$. For $\alpha = \{\alpha_1, \dots, \alpha_m\} \in \tilde{\Gamma}_m$ and for $\pi \in \Pi_m$ we set $\alpha_\pi \in \Gamma_m$ by $\alpha_\pi = (\alpha_{\pi_1}, \dots, \alpha_{\pi_m})$. Finally, for $\alpha \in \tilde{\Gamma}_m$ and $\beta \in \tilde{\Gamma}_m$ we set $\alpha \cup \beta$ and $\alpha \cap \beta$ as the standard reunion and intersection respectively.

Now, $(\alpha, \beta) \in \Lambda_r^{m,n}$ if and only if there exist $\gamma \in \tilde{\Gamma}_r$, $\bar{\alpha} \in \tilde{\Gamma}_{m-r}$, $\bar{\beta} \in \tilde{\Gamma}_{n-r}$, $\pi \in \Pi_m$ and $\sigma \in \Pi_n$ such that $\gamma \cap \bar{\alpha} = \emptyset$, $\gamma \cap \bar{\beta} = \emptyset$, $\bar{\alpha} \cap \bar{\beta} = \emptyset$ and finally, $\alpha = (\bar{\alpha} \cup \gamma)_\pi$ and $\beta = (\bar{\beta} \cup \gamma)_\sigma$. Therefore, by using the

symmetry property for $f_{(m)}$,

$$\begin{aligned} \sum_{(\alpha, \beta) \in \Lambda_r^{m, n}} f_{(m)}(\alpha) f_{(n)}(\beta) Z^\alpha Z^\beta &= \sum_{(\pi, \sigma) \in \Pi_m \times \Pi_n} \sum_{\substack{(\gamma, \bar{\alpha}, \bar{\beta}) \in \tilde{\Gamma}_r \times \tilde{\Gamma}_{m-r} \times \tilde{\Gamma}_{n-r} \\ \gamma \cap \bar{\alpha} = \emptyset, \gamma \cap \bar{\beta} = \emptyset, \bar{\alpha} \cap \bar{\beta} = \emptyset}} f_{(m)}(\bar{\alpha}, \gamma) f_{(n)}(\bar{\beta}, \gamma) Z^{\bar{\alpha}} Z^{\bar{\beta}} (Z^\gamma)^2 \\ &= m!n! \sum_{\gamma \in \tilde{\Gamma}_r} \sum_{\substack{(\bar{\alpha}, \bar{\beta}) \in \tilde{\Gamma}_{m-r} \times \tilde{\Gamma}_{n-r} \\ \bar{\alpha} \cap \bar{\beta} = \emptyset}} f_{(m)}(\bar{\alpha}, \gamma) f_{(n)}(\bar{\beta}, \gamma) Z^{\bar{\alpha}} Z^{\bar{\beta}} (Z^\gamma)^2 \end{aligned}$$

Since $\gamma \mapsto f_{(m)}(\bar{\alpha}, \gamma) f_{(n)}(\bar{\beta}, \gamma) Z^{\bar{\alpha}} Z^{\bar{\beta}} (Z^\gamma)^2$ is symmetric, and similarly for $\bar{\alpha}$ and $\bar{\beta}$, we get

$$\begin{aligned} \sum_{(\alpha, \beta) \in \Lambda_r^{m, n}} f_{(m)}(\alpha) f_{(n)}(\beta) Z^\alpha Z^\beta &= m!n! \frac{1}{r!(m-r)!(n-r)!} \times \\ &\times \sum_{\gamma \in \Gamma_r} \sum_{\eta \in \Gamma_{m-r+n-r}} f_{(m)}(\mathbf{p}_{m-r}(\eta), \gamma) f_{(n)}(\mathbf{q}_{n-r}(\eta), \gamma) Z^\eta (Z^\gamma)^2. \end{aligned}$$

Then, we have

$$\begin{aligned} S(f, Z) &= \sum_{m \geq 0} \sum_{n \geq 0} \sum_{r=0}^{m \wedge n} r! \binom{m}{r} \binom{n}{r} \sum_{\eta \in \Gamma_{m+n-2r}} \sum_{\gamma \in \Gamma_r} f_{(m)}(\mathbf{p}_{m-r}(\eta), \gamma) f_{(n)}(\mathbf{q}_{n-r}(\eta), \gamma) (Z^\gamma)^2 Z^\eta \\ &= \sum_{r \geq 0} r! \sum_{m \geq r} \sum_{n \geq r} \binom{m}{r} \binom{n}{r} \sum_{\eta \in \Gamma_{m+n-2r}} \sum_{\gamma \in \Gamma_r} f_{(m)}(\mathbf{p}_{m-r}(\eta), \gamma) f_{(n)}(\mathbf{q}_{n-r}(\eta), \gamma) (Z^\gamma)^2 Z^\eta \end{aligned}$$

We consider the change of variable $a = m - r$ and $b = m - r + n - r = a + n - r$. We get

$$\begin{aligned} S(f, Z) &= \sum_{r \geq 0} \sum_{a \geq 0} \sum_{b \geq a} r! \binom{a+r}{r} \binom{b-a+r}{r} \sum_{\eta \in \Gamma_b} \sum_{\gamma \in \Gamma_r} f_{(a+r)}(\mathbf{p}_a(\eta), \gamma) f_{(b-a+r)}(\mathbf{q}_{b-a}(\eta), \gamma) (Z^\gamma)^2 Z^\eta \\ &= \sum_{r \geq 0} \sum_{b \geq 0} \sum_{\eta \in \Gamma_b} \sum_{\gamma \in \Gamma_r} \tilde{A}_{b,r}[f](\eta, \gamma) (Z^\gamma)^2 Z^\eta \end{aligned}$$

where, for $\eta \in \Gamma_b$ and $\gamma \in \Gamma_r$,

$$\tilde{A}_{b,r}[f](\eta, \gamma) = r! \sum_{a=0}^b \binom{a+r}{r} \binom{b-a+r}{r} f_{(a+r)}(\mathbf{p}_a(\eta), \gamma) f_{(b-a+r)}(\mathbf{q}_{b-a}(\eta), \gamma).$$

We notice that $\gamma \mapsto \tilde{A}_{b,r}[f](\eta, \gamma)$ is symmetric but $\eta \mapsto \tilde{A}_{b,r}[f](\eta, \gamma)$ is not. So, in order to work with a coefficient $A_{b,r}[f](\eta, \gamma)$ which is (separately) symmetric in both variables η and γ , we use the fact that

$$\sum_{\eta \in \Gamma_b} \tilde{A}_{b,r}[f](\eta, \gamma) Z^\eta = \sum_{\eta \in \Gamma_b} \frac{1}{b!} \sum_{\pi \in \Pi_b} \tilde{A}_{b,r}[f](\eta_\pi, \gamma) Z^\eta$$

where Π_b denotes all the permutations of $(1, \dots, b)$ and for $\pi \in \Pi_b$, $\eta_\pi = (\eta_{\pi_1}, \dots, \eta_{\pi_b})$. Therefore,

$$S(f, Z) = \sum_{r \geq 0} \sum_{b \geq 0} \sum_{\eta \in \Gamma_b} \sum_{\gamma \in \Gamma_r} A_{b,r}[f](\eta, \gamma) (Z^\gamma)^2 Z^\eta$$

and $A_{b,r}[f]$ fulfils formula (B.19). \square

For $n, r \geq 0$, $m \geq r$, $\eta \in \Gamma_n$ and $\rho \in \Gamma_r$ we define

$$\begin{aligned}
B_{n,r,m}[f](\eta, \rho) &= \sum_{\beta \in \Gamma_{m-r}} A_{n,m}[f](\eta, (\rho, \beta)) \\
&= \frac{m!}{n!} \sum_{a=0}^n \binom{a+m}{m} \binom{n-a+m}{m} \sum_{\pi \in \Pi_n} \sum_{\beta \in \Gamma_{m-r}} f_{(a+m)}(\mathfrak{p}_a(\eta_\pi), \rho, \beta) f_{(n-a+m)}(\mathfrak{q}_{n-a}(\eta_\pi), \rho, \beta) \\
&= \frac{m!}{n!} \sum_{a=0}^n \binom{a+m}{m} \binom{n-a+m}{m} \sum_{\pi \in \Pi_n} f_{(a+m)} \otimes_{m-r} f_{(n-a+m)}((\mathfrak{p}_a(\eta_\pi), \rho), (\mathfrak{q}_{n-a}(\eta_\pi), \rho)), \quad (\text{B.21})
\end{aligned}$$

\mathfrak{p}_a and \mathfrak{q}_{n-a} being defined in (B.20). As a consequence, $B_{n,r,m}[f](\eta, \rho) = 0$ if η and ρ have common components and the maps $\eta \mapsto B_{n,r,m}(\eta, \rho)$ and $\rho \mapsto B_{n,r,m}(\eta, \rho)$ are both symmetric.

Lemma B.6 *Let $Y_i = Z_i^2 - 1$, $i \geq 1$ and let $t_{n,r}(\cdot, \cdot)$ be defined in (B.12) with respect to $Z = (Z_i)_{i \in \mathbb{N}}$ and $Y = (Y_i)_{i \in \mathbb{N}}$. We have*

$$|S(f, Z)|^2 = \sum_{r \geq 0} \sum_{m \geq r} \sum_{n \geq 0} \binom{m}{r} t_{n,r}(+\infty, B_{n,r,m}[f]), \quad (\text{B.22})$$

$B_{n,r,m}[f]$ being given in (B.21).

Proof. We start from the equality

$$\prod_{i=1}^m x_i^2 = 1 + \sum_{r=1}^m \sum_{\Lambda = \{\Lambda_1, \dots, \Lambda_r\} \subset \{1, \dots, m\}} \prod_{i=1}^r (x_{\Lambda_i}^2 - 1).$$

Fix now $\gamma \in \Gamma_m$. For a fixed $r = 1, \dots, m$ and $\Lambda = \{\Lambda_1, \dots, \Lambda_r\} \subset \{1, \dots, m\}$, we set $\gamma_\Lambda = (\gamma_{\Lambda_1}, \dots, \gamma_{\Lambda_r})$. Then

$$\begin{aligned}
(Z^\gamma)^2 &= \prod_{i=1}^m Z_{\gamma_i}^2 = 1 + \sum_{r=1}^m \sum_{\Lambda = \{\Lambda_1, \dots, \Lambda_r\} \subset \{1, \dots, m\}} \prod_{i=1}^r (Z_{\gamma_{\Lambda_i}}^2 - 1) \\
&= 1 + \sum_{r=1}^m \sum_{\Lambda = \{\Lambda_1, \dots, \Lambda_r\} \subset \{1, \dots, m\}} Y^{\gamma_\Lambda}.
\end{aligned}$$

Then, for $\eta \in \Gamma_n$, we can write

$$\begin{aligned}
\sum_{\gamma \in \Gamma_m} (Z^\gamma)^2 A_{n,m}[f](\eta, \gamma) &= \sum_{\gamma \in \Gamma_m} A_{n,m}[f](\eta, \gamma) + \sum_{r=1}^m \sum_{\gamma \in \Gamma_m} \sum_{\Lambda = \{\Lambda_1, \dots, \Lambda_r\} \subset \{1, \dots, m\}} A_{n,m}[f](\eta, \gamma) Y^{\gamma_\Lambda} \\
&= \sum_{\gamma \in \Gamma_m} A_{n,m}[f](\eta, \gamma) + \sum_{r=1}^m \sum_{\rho \in \Gamma_r} Y^\rho \binom{m}{r} \sum_{\beta \in \Gamma_{m-r}} A_{n,m}[f](\eta, (\rho, \beta)),
\end{aligned}$$

the last inequality following from the fact that $\gamma \mapsto A_{n,m}[f](\eta, \gamma)$ is symmetric. We also notice that the case $r = 0$ can be easily inserted in the sum of the above r.h.s. (as usual, $\Gamma_0 = \{\emptyset\}$ and $Y^\emptyset = 1$). Then,

$$\sum_{\gamma \in \Gamma_m} (Z^\gamma)^2 A_{n,m}[f](\eta, \gamma) = \sum_{r=0}^m \sum_{\rho \in \Gamma_r} Y^\rho \binom{m}{r} \sum_{\beta \in \Gamma_{m-r}} A_{n,m}[f](\eta, (\rho, \beta)).$$

By inserting in (B.18), we obtain

$$\begin{aligned}
|S(f, Z)|^2 &= \sum_{m \geq 0} \sum_{n \geq 0} \sum_{\eta \in \Gamma_n} Z^\eta \sum_{\gamma \in \Gamma_m} (Z^\gamma)^2 A_{n,m}[f](\eta, (\rho, \beta)) \\
&= \sum_{m \geq 0} \sum_{n \geq 0} \sum_{\eta \in \Gamma_n} Z^\eta \sum_{r=0}^m \sum_{\rho \in \Gamma_r} Y^\rho \binom{m}{r} \sum_{\beta \in \Gamma_{m-r}} A_{n,m}[f](\eta, (\rho, \beta)) \\
&= \sum_{r \geq 0} \sum_{m \geq r} \sum_{n \geq 0} \binom{m}{r} \sum_{\eta \in \Gamma_n} \sum_{\rho \in \Gamma_r} Z^\eta Y^\rho \sum_{\beta \in \Gamma_{m-r}} A_{n,m}[f](\eta, (\rho, \beta)) \\
&= \sum_{r \geq 0} \sum_{m \geq r} \sum_{n \geq 0} \binom{m}{r} t_{n,r}(+\infty, B_{n,r,m}[f]).
\end{aligned}$$

□

We take now $c = (c_{(m)})_{m \geq 1}$, with $c_{(m)} \in \mathcal{H}^{\otimes m}$, $c_{(m)}$ symmetric and null on all diagonals. We set $|c|_m = |c_{(m)}|_m = \|c_{(m)}\|_{\mathcal{H}^{\otimes m}}$. Recall that

$$\Phi_m(m, Z) = \sum_{\alpha \in \Gamma_m} c_{(m)}(\alpha) Z^\alpha \quad \text{and} \quad S(c, Z) = \sum_{m \geq 1} \Phi_m(c_{(m)}, Z).$$

We also set $\partial_j S(c, Z)$ as the derivative of $S(c, Z)$ w.r.t. Z_j and $D_j S(c, Z)$ as the Malliavin derivative in the j th direction. Thus,

$$D_j S(c, Z) = \chi_j \partial_j S(c, Z).$$

We recall that we have to deal with $\|\tilde{I}(c, Z)\|_p$ and $\|I(c, Z) - i(c)\|_p$, where

$$\tilde{I}(c, Z) = \sum_j \tilde{\chi}_j |\partial_j S(c, Z)|^2, \quad I(c, Z) = \sum_j |\partial_j S(c, Z)|^2, \quad i(c) = \sum_{m \geq 1} m! |c|_m^2 = \|S(c, Z)\|_2^2.$$

We also recall that

$$\partial_j S(c, Z) = \sum_{m \geq 0} \Phi_m(\tilde{c}_{(m)}^j, Z) = S(\tilde{c}^j, Z), \quad \text{with} \quad \tilde{c}_{(m)}^j = (1+m)c(\alpha, j) \mathbf{1}_{j \notin \alpha}, \quad \alpha \in \Gamma_m. \quad (\text{B.23})$$

The case $m = 1$ is allowed: just set $\Gamma_0 = \{\emptyset\}$, $Z^\emptyset = 1$ and $c_{(0)}^j = c(j)$. Therefore, we can write

$$\begin{aligned}
\tilde{I}(c, Z) &= \sum_j \tilde{\chi}_j |S(\tilde{c}^j, Z)|^2 \quad \text{and} \quad I(c, Z) = \sum_j |S(\tilde{c}^j, Z)|^2, \\
\text{with} \quad \tilde{c}_{(m)}^j &= (1+m)c(\alpha, j) \mathbf{1}_{j \notin \alpha}, \quad \alpha \in \Gamma_m
\end{aligned} \quad (\text{B.24})$$

We can then write $\tilde{I}(c, Z)$ and $I(c, Z)$ as follows.

Lemma B.7 (i) Let $\tilde{I}(c, Z)$ and $T_{n,r}(\cdot, \cdot)$ (associated to Z_i and $Y_i = Z_i^2 - 1$, $i \in \mathbb{N}$) be defined in (B.24) and (B.13) respectively. Then,

$$\tilde{I}(c, Z) = \sum_{r \geq 0} \sum_{m \geq r} \sum_{n \geq 0} T_{n,r}(+\infty, \tilde{e}_{n,r,m}[c]) \quad (\text{B.25})$$

in which $\tilde{e}_{n,r,m}[c] = (\tilde{e}_{n,r,m}^j[c])_{j \in \mathbb{N}}$ and for $j \in \mathbb{N}$, $\eta \in \Gamma_n$, $\rho \in \Gamma_r$,

$$\begin{aligned}
\tilde{e}_{n,r,m}^j[c](\eta, \rho) &= \frac{(m+1)(m+1)!}{n!} \sum_{a=0}^n \binom{a+m+1}{m+1} \binom{n-a+m+1}{m+1} \times \\
&\times \sum_{\pi \in \Pi_n} c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}((\mathbf{p}_a(\eta_\pi), \rho, j), (\mathbf{q}_{n-a}(\eta_\pi), \rho, j)).
\end{aligned} \quad (\text{B.26})$$

As a consequence, $\tilde{e}_{n,r,m}^j[c](\eta, \rho) = 0$ if η and ρ have common components or if $j \in (\eta, \rho)$. And the maps $\eta \mapsto \tilde{e}_{n,r,m}^j[c](\eta, \rho)$ and $\rho \mapsto \tilde{e}_{n,r,m}^j[c](\eta, \rho)$ are both symmetric.

(ii) Let $I(c, Z)$ and $t_{n,r}(\cdot, \cdot)$ (associated to Z_i and $Y_i = Z_i^2 - 1$, $i \in \mathbb{N}$) be defined in (B.24) and (B.12) respectively. Then,

$$I(c, Z) = \sum_{r \geq 0} \sum_{m \geq r} \sum_{n \geq 0} t_{n,r}(+\infty, e_{n,r,m}[c]) \quad (\text{B.27})$$

in which, for $\eta \in \Gamma_n$ and $\rho \in \Gamma_r$,

$$\begin{aligned} e_{n,r,m}[c](\eta, \rho) &= \frac{(m+1)(m+1)!}{n!(m-r+1)!} \sum_{a=0}^n \binom{a+m+1}{m+1} \binom{n-a+m+1}{m+1} \times \\ &\times \sum_{\pi \in \Pi_n} c_{(a+m+1)} \otimes_{m-r+1} c_{(n-a+m+1)}((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)). \end{aligned} \quad (\text{B.28})$$

As a consequence, $e_{n,r,m}[c](\eta, \rho) = 0$ if η and ρ have common components and the maps $\eta \mapsto e_{n,r,m}[c](\eta, \rho)$ and $\rho \mapsto e_{n,r,m}[c](\eta, \rho)$ are both symmetric.

Proof. (i) By (B.24), we use Lemma B.6 with $f_{(m)} = \tilde{c}_{(m)}^j$ and we obtain

$$\begin{aligned} \tilde{I}(c, Z) &= \sum_j \tilde{\chi}_j |S(\hat{c}^j, Z)|^2 \\ &= \sum_j \tilde{\chi}_j \sum_{r \geq 0} \sum_{m \geq r} \sum_{n \geq 0} \binom{m}{r} t_{n,r}(+\infty, B_{n,r,m}[\hat{c}^j]). \end{aligned}$$

We develop $t_{n,r}(+\infty, B_{n,r,m}[\hat{c}^j])$ according to (B.12) and we find that the coefficient of $Z^n Y^r \tilde{\chi}_j$ is $B_{n,r,m}[\hat{c}^j](\eta, \rho)$. And by definition we have denoted this quantity by $\tilde{e}_{n,r,m}^j[c](\eta, \rho)$. Using now the expression given in (B.21) we obtain

$$\tilde{e}_{n,r,m}^j[c](\eta, \rho) = \frac{m!}{n!} \sum_{a=0}^n \binom{a+m}{m} \binom{n-a+m}{m} \sum_{\pi \in \Pi_n} \tilde{c}_{(a+m)}^j \otimes_{m-r} \tilde{c}_{(n-a+m)}^j((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)).$$

Since $\tilde{c}_{(m)}^j(\alpha) = (1+m)c_{(1+m)}(\alpha, j)$, we get

$$\begin{aligned} \tilde{c}_{(a+m)}^j \otimes_{m-r} \tilde{c}_{(n-a+m)}^j((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)) &= \\ &= (a+m+1)(n-a+m+1)c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}((\mathbf{p}_a(\eta_\pi), \rho, j), (\mathbf{q}_{n-a}(\eta_\pi), \rho, j)). \end{aligned} \quad (\text{B.29})$$

So,

$$\begin{aligned} \tilde{e}_{n,r,m}^j[c](\eta, \rho) &= \frac{m!}{n!} \sum_{a=0}^n \binom{a+m}{m} \binom{n-a+m}{m} (a+m+1)(n-a+m+1) \times \\ &\times \sum_{\pi \in \Pi_n} c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}((\mathbf{p}_a(\eta_\pi), \rho, j), (\mathbf{q}_{n-a}(\eta_\pi), \rho, j)) \\ &= \frac{(m+1)(m+1)!}{n!} \sum_{a=0}^n \binom{a+m+1}{m+1} \binom{n-a+m+1}{m+1} \times \\ &\times \sum_{\pi \in \Pi_n} c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}((\mathbf{p}_a(\eta_\pi), \rho, j), (\mathbf{q}_{n-a}(\eta_\pi), \rho, j)) \end{aligned}$$

and the proof of (i) is completed.

(ii) By (B.24), we use Lemma B.6 with $f_{(m)} = \tilde{c}_{(m)}^j$ and we have the result with $e_{n,r,m}[c](\eta, \rho) = \sum_{j \geq 1} B_{n,r,m}[\hat{c}^j](\eta, \rho)$. By inserting formula (B.21), we obtain

$$e_{n,r,m}[c](\eta, \rho) = \sum_{j \geq 1} \frac{m!}{n!} \sum_{a=0}^n \binom{a+m}{m} \binom{n-a+m}{m} \sum_{\pi \in \Pi_n} \tilde{c}_{(a+m)}^j \otimes_{m-r} \tilde{c}_{(n-a+m)}^j ((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)).$$

We use now (B.29) and we get

$$\begin{aligned} & \sum_{j \geq 1} \tilde{c}_{(a+m)}^j \otimes_{m-r} \tilde{c}_{(n-a+m)}^j ((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)) = \\ &= (a+m+1)(n-a+m+1) \sum_{j \geq 1} c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)} ((\mathbf{p}_a(\eta_\pi), \rho, j), (\mathbf{q}_{n-a}(\eta_\pi), \rho, j)) \\ &= \frac{(a+m+1)(n-a+m+1)}{m-r+1} c_{(a+m+1)} \otimes_{m-r+1} c_{(n-a+m+1)} ((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)). \end{aligned}$$

Then,

$$\begin{aligned} e_{n,r,m}[c](\eta, \rho) &= \frac{m!}{n!(m-r+1)} \sum_{a=0}^n \binom{a+m}{m} \binom{n-a+m}{m} (a+m+1)(n-a+m+1) \times \\ & \quad \times \sum_{\pi \in \Pi_n} c_{(a+m+1)} \otimes_{m-r+1} c_{(n-a+m+1)} ((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)) \\ &= \frac{(m+1)(m+1)!}{n!(m-r+1)} \sum_{a=0}^n \binom{a+m+1}{m+1} \binom{n-a+m+1}{m+1} \times \\ & \quad \times \sum_{\pi \in \Pi_n} c_{(a+m+1)} \otimes_{m-r+1} c_{(n-a+m+1)} ((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho)) \end{aligned}$$

□

B.3 L^p estimates

We can now use Lemma B.3 and finally state the estimate for the L^p norms of $\tilde{I}(c, Z)$ and $I(c, Z) - i(c)$. Here we are obliged to restrict ourselves to finite series, so we fix N and we suppose that

$$c(\alpha) = 0 \quad \text{for} \quad |\alpha| > N.$$

In this case we denote $\tilde{I}_N(c, Z)$, $I_N(c, Z)$ and $i_N(c)$ instead of $\tilde{I}(c, Z)$, $I(c, Z)$ and $i(c)$. We also recall that $\kappa_{4,\ell}(c)$ is the 4th cumulant given in (2.10) and $M_p(Z) = 1 \vee \sup_k \|Z_k\|_p$.

Lemma B.8 *Fore each $p \geq 1$ there exists a universal constant $C_p \geq 1$ such that*

$$\begin{aligned} \|\tilde{I}_N(c, Z)\|_p + \|I_N(c, Z) - i_N(c)\|_p &\leq C_p (1 + i_N(c))^{1/2} (N!)^3 2^N N^{-5/4} \times \\ & \quad \times \left[\sum_{l=1}^N \kappa_{4,l}(c)^{1/4} + \sum_{l=1}^N \delta_l(c) \right], \end{aligned} \tag{B.30}$$

where $C_p > 0$ is a constant depending on p only.

Proof. Step 1: estimate of $\tilde{I}_N(c, Z)$. We recall the expression (B.25) for $\tilde{I}_N(c, Z)$, based on the coefficients $\tilde{e}_{n,r,m}^j[c](\eta, \rho) = (\tilde{e}_{n,r,m}^j[c](\eta, \rho))_j$ given in (B.26). We notice that $\tilde{e}_{n,r,m}^j[c] \equiv 0$ if, for every $a = 0, 1, \dots, n$, one has $c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)} \equiv 0$, and this latter property holds if $a+m+1 > N$ or $n-a+m+1 > N$. Then, $\tilde{e}_{n,r,m}^j[c] \equiv 0$ if $m+1 > N$.

Moreover, for $a = 0, 1, \dots, n$ we have

$$\begin{aligned} \binom{a+m+1}{m+1} \binom{n-a+m+1}{m+1} &= \frac{(a+m+1)!}{(m+1)!} \frac{(n-a+m+1)!}{(m+1)!} \binom{n}{a} \frac{1}{n!} \\ &\leq \frac{2^n}{n!} \frac{(a+m+1)!}{(m+1)!} \frac{(n-a+m+1)!}{(m+1)!}. \end{aligned}$$

Recall that in $\tilde{e}_{n,r,m}^j[c](\eta, \rho)$ this term multiplies $c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}$ which is null for $a+m+1 > N$ or $n-a+m+1 > N$. So, we consider only the terms with a such that $a+m+1 \leq N$ and $n-a+m+1 \leq N$, and notice that this gives the following request:

$$m+1 \leq N \quad \text{and} \quad n \leq 2(N-m-1). \quad (\text{B.31})$$

So, we can write

$$\binom{a+m+1}{m+1} \binom{n-a+m+1}{m+1} \leq \frac{2^n}{n!} \frac{(N!)^2}{((m+1)!)^2}. \quad (\text{B.32})$$

Then,

$$|\tilde{e}_{n,r,m}^j[c](\eta, \rho)| \leq \frac{2^n (N!)^2}{(n!)^2 m!} \sum_{a=0}^n \sum_{\pi \in \Pi_n} |c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}((\mathfrak{p}_a(\eta_\pi), \rho, j), (\mathfrak{q}_{n-a}(\eta_\pi), \rho, j))|,$$

thereby

$$\begin{aligned} &|\tilde{e}_{n,r,m}^j[c](\eta, \rho)|^2 \\ &\leq \left(\frac{2^n (N!)^2}{(n!)^2 m!} \right)^2 \times n \times n! \sum_{a=0}^n \sum_{\pi \in \Pi_n} |c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}((\mathfrak{p}_a(\eta_\pi), \rho, j), (\mathfrak{q}_{n-a}(\eta_\pi), \rho, j))|^2. \end{aligned} \quad (\text{B.33})$$

Now we have to distinguish two cases: $m \geq r+1$ and $m = r$. We assume first that $m \geq r+1$ and we use (B.6) in order to obtain

$$\begin{aligned} |\tilde{e}_{n,r,m}[c]|_{n,r,\infty}^2 &= \sum_{j \geq 1} \sum_{\eta \in \Gamma_n} \sum_{\rho \in \Gamma_r} |\tilde{e}_{n,r,m}^j[c](\eta, \rho)|^2 \\ &\leq \left(\frac{2^n (N!)^2}{(n!)^2 m!} \right)^2 \times n \times n! \sum_{a=0}^n |c_{(a+m+1)} \otimes_{m-r} c_{(n-a+m+1)}|^2 \\ &\leq \left(\frac{2^n (N!)^2}{(n!)^2 m!} \right)^2 \times n \times n! 2 \sum_{a=0}^n \frac{\kappa_{4,a+m+1}(c)}{((a+m+1)!)^2 \binom{a+m+1}{m-r}^2} \end{aligned}$$

so that

$$|\tilde{e}_{n,r,m}[c]|_{n,r,\infty}^2 \leq \left(\frac{2^n (N!)^2}{(n!)^2 m!} \right)^2 \times n \times n! \times \frac{2}{((m+1)!)^2} \sum_{l=1}^N \kappa_{4,l}(c). \quad (\text{B.34})$$

If instead $m = r$, (B.33) gives

$$\begin{aligned} |\tilde{e}_{n,r,r}[c]|_{n,r,\infty}^2 &\leq \left(\frac{2^n (N!)^2}{(n!)^2 r!} \right)^2 \times n \times n! \times \\ &\times \sum_{a=0}^n \sum_{\pi \in \Pi_n} \sum_{j \geq 1} \sum_{\eta \in \Gamma_n} \sum_{\rho \in \Gamma_r} |c_{(a+r+1)}(\mathfrak{p}_a(\eta_\pi), \rho, j) c_{(n-a+r+1)}(\mathfrak{q}_{n-a}(\eta_\pi), \rho, j)|^2 \\ &\leq \left(\frac{2^n (N!)^2}{(n!)^2 r!} \right)^2 \times n \times n! \times \sum_{a=0}^n \delta_{a+r+1}^2(c) |c|_{n-a+r+1}^2 \end{aligned}$$

and we obtain

$$|\tilde{e}_{n,r,r}[c]|_{n,r,\infty}^2 \leq \left(\frac{2^n (N!)^2}{(n!)^2 r!} \right)^2 \times n \times n! \times i_N(c) \sum_{l=1}^N \delta_l^2(c). \quad (\text{B.35})$$

Therefore, by (B.25), recalling the conditions ((B.31)), and by using (B.15), we can write

$$\begin{aligned} \|\tilde{I}(c, Z)\|_p &\leq \sum_{r=0}^{N-1} \sum_{m=r}^{N-1} \sum_{n=0}^{2(N-m-1)} \|T_{n,r}(+\infty, \tilde{e}_{n,r,m}[c])\|_p \\ &\leq \sum_{r=0}^{N-1} \sum_{m=r}^{N-1} \sum_{n=0}^{2(N-m-1)} C_p (4^{n+r} (n+r)!)^{1/2} (\sqrt{2} b_p M_p)^{n+r} |\tilde{e}_{n,r,m}[c]|_{n,r,\infty}, \end{aligned}$$

in which C_p is a constant depending on p only (which may vary in next lines) and

$$M_p = M_p(Z, Y) = \sup_k \|Z_k\|_p \vee \|Y_k\|_p \leq 2 \sup_k \|Z_k\|_{2p}^2 =: A_p.$$

We now split the cases $m > r + 1$ and $m = r$ and we use the estimates (B.34) and (B.35). Then

$$\begin{aligned} \|\tilde{I}(c, Z)\|_p &\leq C_p \sum_{r=0}^{N-2} \sum_{m=r+1}^{N-1} \sum_{n=0}^{2(N-m-1)} (4^{n+r} (n+r)!)^{1/2} (\sqrt{2} b_p A_p)^{n+r} \frac{2^n (N!)^2}{(n!)^2 m!} \sqrt{n n!} \frac{1}{(m+1)!} \left(\sum_{l=1}^N \kappa_{4,l}(c) \right)^{1/2} \\ &\quad + C_p \sum_{r=0}^{N-1} \sum_{n=0}^{2(N-r-1)} (4^{n+r} (n+r)!)^{1/2} (\sqrt{2} b_p A_p)^{n+r} \frac{2^n (N!)^2}{(n!)^2 r!} \sqrt{n n!} \left(i_N(c) \sum_{l=1}^N \delta_l^2(c) \right)^{1/2} \\ &\leq C_p \sum_{r=0}^{N-1} \sum_{n=0}^{2(N-r-1)} (4^{n+r} (n+r)!)^{1/2} (\sqrt{2} b_p A_p)^{n+r} \frac{2^n (N!)^2}{(n!)^2 r!} \sqrt{n n!} \times \\ &\quad \times \left[\sum_{l=1}^N \kappa_{4,l}(c)^{1/2} + \sqrt{i_N(c)} \sum_{l=1}^N \delta_l(c) \right] \\ &\leq C_p (N!)^2 ((2N-2)!)^{1/2} \left[\sum_{l=1}^N \kappa_{4,l}(c)^{1/2} + \sqrt{i_N(c)} \sum_{l=1}^N \delta_l(c) \right], \end{aligned}$$

in which we have used the fact that $(n+r)! \leq (2N-2)!$ for r and n in the range of the above series.

Step 2: estimate of $I_N(c, Z) - i_N(c)$. We recall the expression (B.27) for $I_N(c, Z)$, based on the coefficients $e_{n,r,m}[c](\eta, \rho)$ given in (B.28). We notice that the term $i_N(c)$ is actually the term in the series (B.27) when one takes $n = r = 0$, so that

$$I_N(c, Z) - i_N(c) = \sum_{r \geq 0} \sum_{m \geq r} \sum_{n \geq 0 \vee (1-r)} t_{n,r}(+\infty, e_{n,r,m}[c])$$

Following the same arguments developed in Step 1, we can say that $e_{n,r,m}[c] \equiv 0$ if $m+1 > N$ and the constraints in (B.31) hold. And by using (B.32), we obtain

$$|e_{n,r,m}[c](\eta, \rho)| \leq \frac{2^n (N!)^2}{(n!)^2 m! (m-r+1)} \sum_{a=0}^n \sum_{\pi \in \Pi_n} |c_{(a+m+1)} \otimes_{m-r+1} c_{(n-a+m+1)}((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho))|,$$

so that

$$\begin{aligned} &|e_{n,r,m}[c](\eta, \rho)|^2 \\ &\leq \left(\frac{2^n (N!)^2}{(n!)^2 m! (m-r+1)} \right)^2 \times n \times n! \sum_{a=0}^n \sum_{\pi \in \Pi_n} |c_{(a+m+1)} \otimes_{m-r+1} c_{(n-a+m+1)}((\mathbf{p}_a(\eta_\pi), \rho), (\mathbf{q}_{n-a}(\eta_\pi), \rho))|^2. \end{aligned} \quad (\text{B.36})$$

We have now to split the case $r > 0$ and $r = 0$. For $r > 0$, we use (B.6) and, similarly as before, we obtain

$$|e_{n,r,m}[c]|_{n,r,\infty}^2 \leq \left(\frac{2^n(N!)^2}{(n!)^2 m!(m-r+1)} \right)^2 \times n \times n! \times \frac{2}{((m+1)!)^2} \sum_{l=m+1}^{n+m+1} \kappa_{4,l}(c).$$

When $r = 0$ (recall that in this case $n \geq 1$) we have a different behavior in the sum as $a = 0, \dots, n$. In fact, for $a = 1, \dots, n-1$ we use again (B.6) and we obtain the same estimate as before. But for $a = 0$ and $a = n$, we cannot use (B.6) but we can use (B.7). So, we obtain

$$\begin{aligned} |e_{n,0,m}[c]|_{n,0,\infty}^2 &\leq \left(\frac{2^n(N!)^2}{(n!)^2 m!(m+1)} \right)^2 \times n \times n! \times \frac{2}{((m+1)!)^2} \sum_{l=m+2}^{n+m} \kappa_{4,l}(c) + \\ &\quad + \left(\frac{2^n(N!)^2}{(n!)^2 m!(m+1)} \right)^2 \times n \times n! \times 2|c_{(m+1)} \otimes_{m+1} c_{(n+m+1)}|^2 \\ &\leq \left(\frac{2^n(N!)^2}{(n!)^2 m!(m+1)} \right)^2 \times n \times n! \left(\frac{2}{((m+1)!)^2} \sum_{l=m+2}^{n+m} \kappa_{4,l}(c) + \frac{2}{(m+1)!} |c|_{m+1}^2 \kappa_{4,n+m+1}(c)^{1/2} \right) \\ &\leq 2(1 + i_N(c)) \left(\frac{2^n(N!)^2}{(n!)^2 m!(m+1)} \right)^2 \times \frac{n n!}{(m+1)!} \sum_{l=m+1}^{n+m+1} \kappa_{4,l}(c)^{1/2} \end{aligned}$$

By resuming, for $n, r \geq 0$ such that $n + r \geq 1$ we have

$$|e_{n,r,m}[c]|_{n,r,\infty}^2 \leq 2(1 + i_N(c)) \left(\frac{2^n(N!)^2}{(n!)^2 m!(m-r+1)} \right)^2 \frac{2}{n n! ((m+1)!)^2} \sum_{l=1}^N \kappa_{4,l}(c)^{1/2}.$$

Therefore, by (B.27), recalling the conditions (B.31), and by using (B.14) and the estimate $M_p(Z, Y) \leq A_p$, we can write

$$\begin{aligned} \|I_N(c, Z) - i_N(c)\|_p &\leq \sum_{r=0}^{N-1} \sum_{m=r}^{N-1} \sum_{n=0}^{2(N-m-1)} \mathbf{1}_{n+r \geq 1} \|t_{n,r}(+\infty, e_{n,r,m}[c])\|_p \\ &\leq \sum_{r=0}^{N-1} \sum_{m=r}^{N-1} \sum_{n=0}^{2(N-m-1)} \mathbf{1}_{n+r \geq 1} ((n+r)!)^{1/2} (\sqrt{2} b_p A_p)^{(r+n)} |e_{n,r,m}[c]|_{n,r,\infty} \\ &\leq \sum_{r=0}^{N-1} \sum_{m=r}^{N-1} \sum_{n=0}^{2(N-m-1)} ((n+r)!)^{1/2} (\sqrt{2} b_p A_p)^{(r+n)} \frac{2(1 + i_N(c))^{1/2} 2^n (N!)^2 \sqrt{n n!}}{(n!)^2 m!(m-r+1)(m+1)!} \sum_{l=1}^N \kappa_{4,l}(c)^{1/4} \\ &\leq C_p (1 + i_N(c))^{1/2} (N!)^2 \sum_{l=1}^N \kappa_{4,l}(c)^{1/4} \times \mathcal{S} \end{aligned}$$

where \mathcal{S} is a sum which has a behavior similar to the one studied in step 1. So,

$$\|I_N(c, Z) - i_N(c)\|_p \leq C_p (N!)^2 ((2N-2)!)^{1/2} \sum_{l=1}^N \kappa_{4,l}(c)^{1/4}.$$

The Stirling's approximation formula now gives

$$\exists \lim_{N \rightarrow \infty} \frac{((2N-2)!)^{1/2}}{2^N N^{-5/4} N!} \in (0, 1)$$

so $((2N-2)!)^{1/2} \leq 2^N N^{-5/4} N!$ and the statement finally holds. \square

References

- [1] Bally, V., and Caramellino, L.: Asymptotic development for the CLT in total variation distance. *Bernoulli*, to appear (2015).
- [2] Bally, V., and Caramellino, L.: On the distances between probability density functions. *Electronic Journal of Probability*, **19**, no. 110, 1-33 (2014).
- [3] Bally, V., and Caramellino, L.: An Invariance principle for Stochastic Series II. Non Gaussian limits. Working paper.
- [4] Bally, V., and Clément, E. : Integration by parts formula and applications to equations with jumps. *Probab. Theory Related Fields*, **151**, 613–657 (2011).
- [5] Bakry, D., Gentil, I., and Ledoux, M.: *Analysis and Geometry of Markov Diffusion Semigroups*. Springer (2014)
- [6] Bichtler, K., Gravereaux, J.-B., and Jacod J.: *Malliavin calculus for processes with jumps*. Gordon and Breach Science Publishers (1987).
- [7] de Jong, P.: A central limit theorem for generalized quadratic forms. *Probab. Th. Rel. Fields* **75**, 261-277 (1987).
- [8] de Jong, P.: A central limit theorem for generalized multilinear forms. *Journal of Multivariate Analysis* **34**, 275-289 (1990).
- [9] Hu, Y., and Nualart, D.: *Renormalize Self-intersection Local Time for Fractional Brownian Motion*. Ann. Probab. **33**, Nr 3, 948 - 983 (2005)
- [10] Koroljuk, V. S., and Borovskich, Yu. V.: *Theory of U-statistics*. Mathematics and its Applications, 273. Kluwer Academic Publishers Group, Dordrecht, 1994.
- [11] Ikeda, N., and Watanabe, S.: *Stochastic Differential Equations and Diffusion processes*. North-Holland Mathematical Library 24 (1989).
- [12] Latala, R.: Estimates of moments and Tails of Gaussian chaoses. *Ann. Probab.* **34**, No 6, 2315-2331 (2006).
- [13] Malicet, D., and Poly, G.: Properties of convergence in Dirichlet structures. *J. Funct. Anal.* **264**, 2077–2096 (2013).
- [14] Mossel, E., O'Donnell, R., and Oleszkiewicz, K.: Noise stability of functions with low influences: Invariance and optimality. *Ann. Math.* **171**, pp. 295-341 (2010).
- [15] Noredine, S., and Nourdin, I.: On the Gaussian approximation of vector-valued multiple integrals. *J. Multiv. Anal.* **102**, no. 6, 1008-1017 (2011).
- [16] Nourdin, I., and Peccati, G.: *Normal Approximations Using Malliavin Calculus: from Stein's Method to Universality*. Cambridge Tracts in Mathematics, 192 (2012).
- [17] Nourdin, I., and Peccati, G.: *Stein's method on Wiener chaos*. Probab. Theory Related Fields 145, no. 1, 75-118 (2009).
- [18] Nourdin, I., Peccati, G., and Reinert, G.: Invariance principles for homogeneous sums: universality of Wiener chaos. *Ann. Probab.* **38**, no. 5, 1947-1985 (2010).
- [19] Nourdin, I., Peccati, G., and Réveillac, A.: Multivariate normal approximation using Stein's method and Malliavin calculus. *Ann. Inst. H. Poincaré Probab. Statist.* **46**, no. 1, 45-58 (2010).

- [20] Nourdin, I., and Poly, G.: Convergence in total variation on Wiener chaos. *Stochastic Process. Appl.* **123**, 651–674 (2013).
- [21] Nualart, D.: *The Malliavin calculus and related topics. Second Edition.* Springer-Verlag (2006).
- [22] Nualart, D., and Ortiz-Latorre, S.: Central limit theorem for multiple stochastic integrals and Malliavin calculus. *Stoch. Processes Appl.* **118**, 614-628 (2008).
- [23] Nualart, D., and Peccati, G.: Central limit theorems for sequences of multiple stochastic integrals. *Annals of Probability* **33**, 177-193 (2005).
- [24] Peccati, G., and Tudor, C.A.: Gaussian limits for vector-valued multiple stochastic integrals. *Séminaire de Probabilités XXXVIII*, 247-262 (2004).
- [25] Prohorov, Y.: On a local limit theorem for densities. Doklady Akad. Nauk SSSR (N.S.)83, 797-800 (1952). In Russian.